

Textbook multigrid efficiency for hydrostatic ice flow

Jed Brown¹ Barry Smith²

¹VAW, ETH Zürich

²MCS, Argonne National Lab

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Hydrostatic equations

- ▶ Valid in the limit $w_x \ll u_z$, independent of basal friction¹
- ▶ Eliminate p and w by incompressibility:

3D elliptic system for $\mathbf{u} = (u, v)$

$$-\nabla \cdot \left[\eta \begin{pmatrix} 4u_x + 2v_y & u_y + v_x & u_z \\ u_y + v_x & 2u_x + 4v_y & v_z \end{pmatrix} \right] + \rho g \nabla s = 0$$

$$\eta(\gamma) = \frac{B}{2} (\epsilon^2 + \gamma)^{\frac{1-n}{2n}}, \quad n \approx 3$$

$$\gamma = u_x^2 + v_y^2 + u_x v_y + \frac{1}{4} (u_y + v_x)^2 + \frac{1}{4} u_z^2 + \frac{1}{4} v_z^2$$

and slip boundary $\sigma \cdot \mathbf{n} = \beta^2 \mathbf{u}$ where

$$\beta^2(\gamma_b) = \beta_0^2 (\epsilon_b^2 + \gamma_b)^{\frac{m-1}{2}}, \quad 0 < m \leq 1$$

$$\gamma_b = \frac{1}{2} (u^2 + v^2)$$

¹C. Schoof and R. Hindmarsh, *Thin-film flows with wall slip: an asymptotic analysis of higher order glacier flow models*, 2010

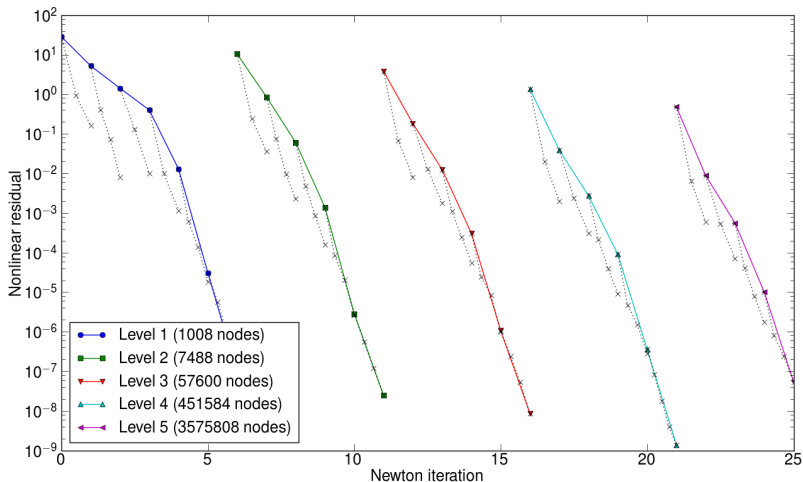
Hydrostatic solver

- ▶ 3D elliptic system for (u, v)

$$-\nabla \cdot \left[\eta \begin{pmatrix} 4u_x + 2v_y & u_y + v_x & u_z \\ u_y + v_x & 2u_x + 4v_y & v_z \end{pmatrix} \right] + \rho g \nabla s = 0$$

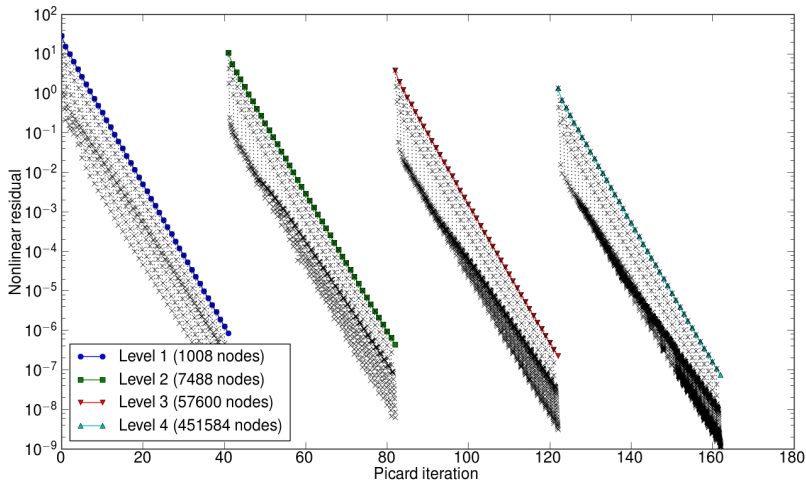
- ▶ strong anisotropy and viscosity variation, shear important
- ▶ current solvers
 - ▶ Picard iteration
 - ▶ serial linear solve or 1-level ASM, high iteration counts
 - ▶ slow convergence with nonlinear sliding
- ▶ `petsc/src/snes/examples/tutorials/ex48.c`
 - ▶ conforming Q_1 finite element discretization, no σ -transform
 - ▶ multigrid Newton-Krylov
 - ▶ smoother respects strong coupling in vertical: rapid coarsening
 - ▶ globalized by grid sequencing
 - ▶ quadratic convergence including slip conditions
 - ▶ robust to discontinuous sliding, high aspect ratio
 - ▶ high throughput: SSE2 integration, blocked matrix formats

Convergence with grid sequencing



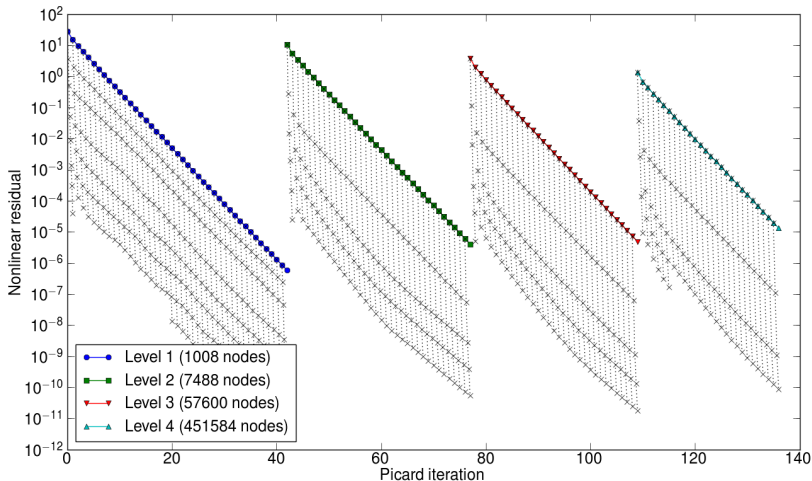
- ▶ colored points are nonlinear residuals
- ▶ black \times marks are unpreconditioned linear residuals

Picard with ASM overlap 1/ICC(1), 8 subdomains



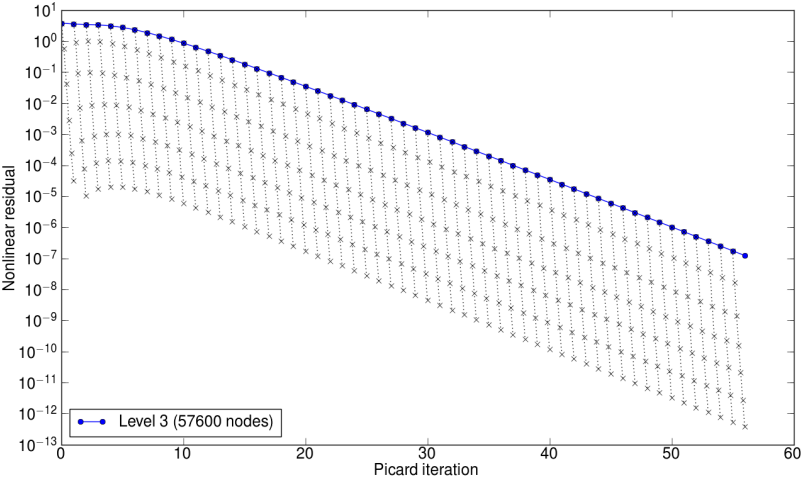
- ▶ $n = 3, m = 1/3, \frac{\eta_{\max}}{\eta_{\min}} \approx 380$
- ▶ geometry of ISMIP-HOM test A, “dimpled sombrero” sliding
- ▶ slow nonlinear convergence, over 100 iterations for $\text{rtol } 10^{-2}$

Picard with multigrid preconditioning, 512 subdomains



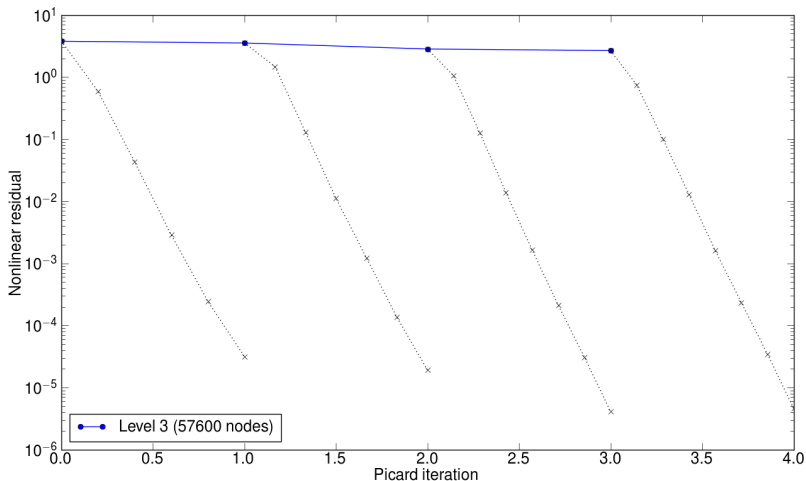
- ▶ slow nonlinear convergence
- ▶ very fast linear solves (note rtol is now 10^{-5})

Picard without grid sequencing



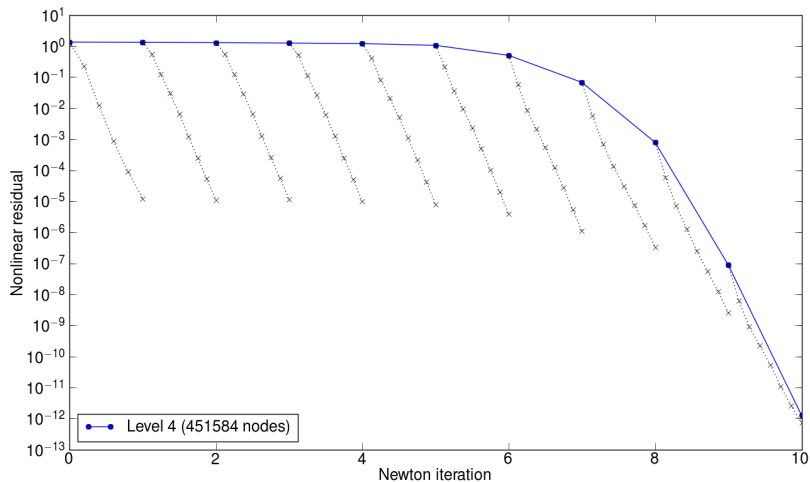
► slow initial convergence

Picard for rheology, Newton-linearized sliding



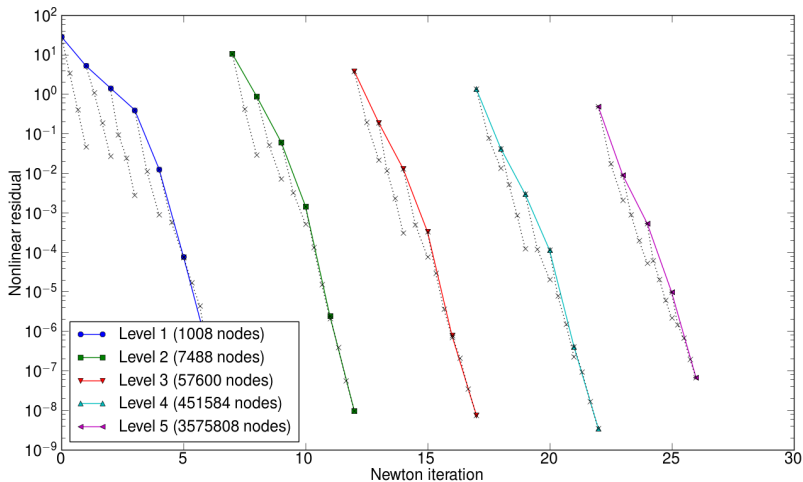
- ▶ cannot Newton-linearize only sliding, step is not descent direction

Switch to Newton



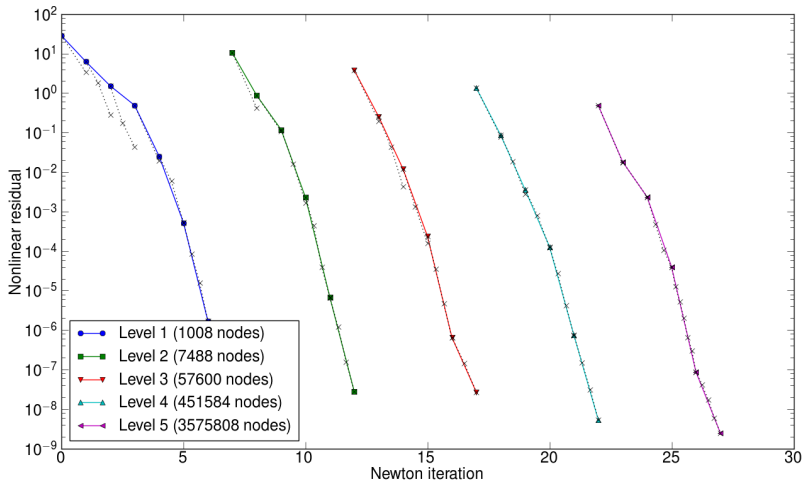
- ▶ *much* faster nonlinear convergence
- ▶ linear systems slightly more difficult
- ▶ initial nonlinear convergence is slow

Nonlinear convergence



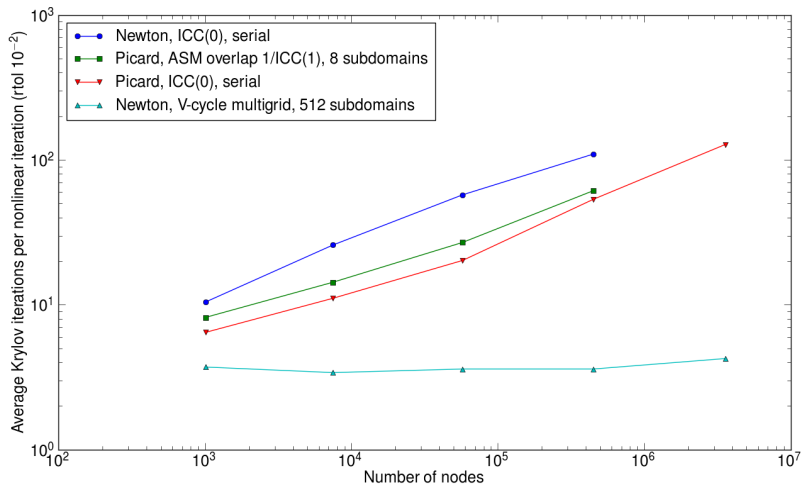
- ▶ ASM/ICC(0) smoothers, isotropic coarsening
- ▶ 512 subdomains on fine levels

Avoid oversolving



- ▶ Luis Chacon's variant of Eisenstat-Walker
- ▶ almost equivalent nonlinear convergence

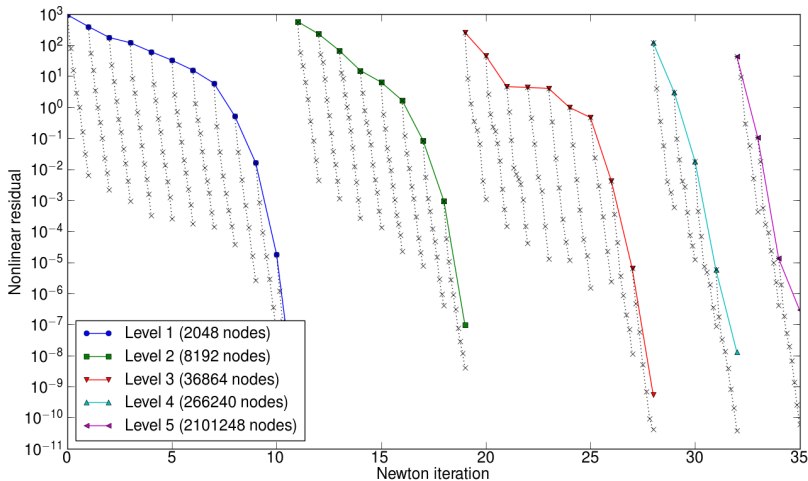
Linear solve performance



Aggressive coarsening

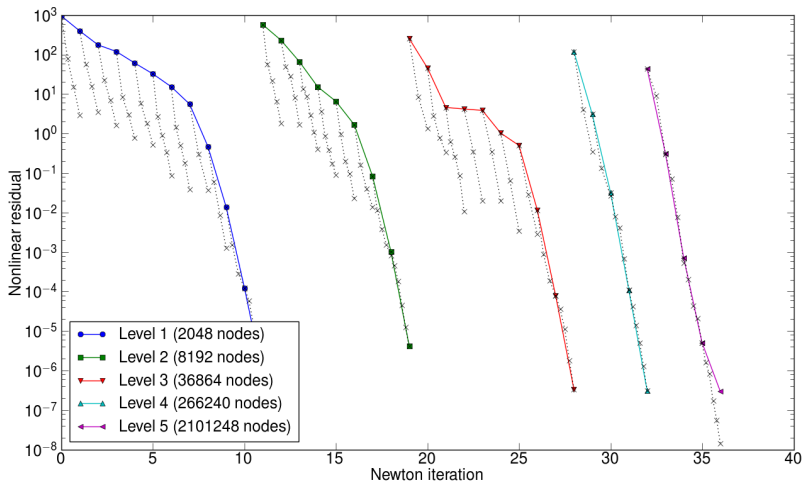
- ▶ coarsest level is responsible for providing global coupling
- ▶ subdomains very small on coarse levels: bottleneck is latency
- ▶ approaches to anisotropy
 - ▶ semi-coarsening (z -only): many intermediate levels, expensive
 - ▶ line smothers: allow rapid coarsening
- ▶ order unknowns so that ICC(0) performs exact column solves
 - ▶ allows isotropic coarsening at moderate aspect ratios
 - ▶ allows semicoarsening by factors ≥ 8

Nonlinear convergence: $n = 3, m = 1, \eta_{\max}/\eta_{\min} = 3800$



- ▶ element aspect ratio 640
- ▶ discontinuous sliding, strong shear zone (0.5 m/a to 46 km/a)
- ▶ rapid semi-coarsening, then isotropic quasi-2D coarsening

Same problem, $\text{rtol } 10^{-2}$



► almost equivalent convergence

Breakdown of incomplete factorization

- ▶ Jacobian is symmetric positive definite
- ▶ full Cholesky always has positive pivots
- ▶ incomplete Cholesky can generate negative pivots
- ▶ increasing fill can make the problem worse
- ▶ standard practice is to “shift” diagonal
 - ▶ may still be an adequate local smoother
 - ▶ destroys nonlocal properties of preconditioner
 - ▶ increasing overlap often makes it worse
- ▶ additive Schwarz with incomplete Cholesky is unreliable
- ▶ additive Schwarz with direct solves is reliable but slow

Conclusions

- ▶ Newton is almost 10 times faster than Picard
- ▶ deficiencies of serial incomplete factorization is more evident with Newton than Picard
- ▶ incomplete factorization is poor with 1-level domain decomposition
- ▶ multigrid preconditioning is ~ 50 times faster at $\sim 1M$ nodes
- ▶ grid sequencing is important for globalization
- ▶ preconditioner can be lagged with true Jacobian applied matrix-free