Improving the numerical convergence of viscous-plastic sea ice models with the Jacobian-free Newton-Krylov method

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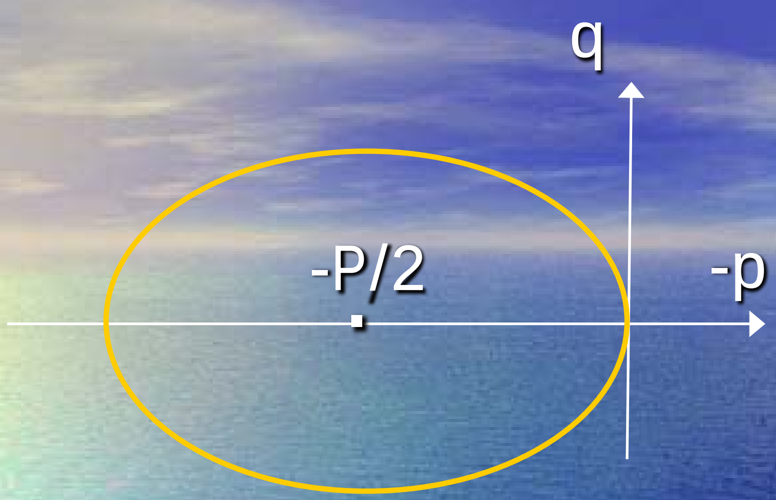
The model

- Dynamic/thermodynamic
- VP rheology, ellipse (Hibler, 1979)
- Domain: Arctic, North Atlantic and CAA
- Resolutions: 10, 20, 40 or 80 km
- Coupled to a slab ocean model
- Forcing:

  - geostrophic winds NCEP/NCAR (6h)

  - climatological currents

Viscous-plastic formulation



$$\sigma_{ij} = 2\eta \dot{\varepsilon}_{ij} + [\zeta - \eta]\dot{\varepsilon}_{kk}\,\delta_{ij} - P\delta_{ij}/2$$

$$i, j = 1,2$$

$$\zeta = P/2\Delta,\ \eta = \zeta e^{-2}$$

$$\Delta = \sqrt{f(\dot{\varepsilon}_{ij}^{2})}$$

Hibler, 1979

$$\Rightarrow \zeta = \min(P/2\Delta, \zeta_{\max})$$

The standard solver

At time t, we want to solve
the nonlinear system of equations:

$$F(u)=A(u)u-b=0$$

u^0: initial iterate

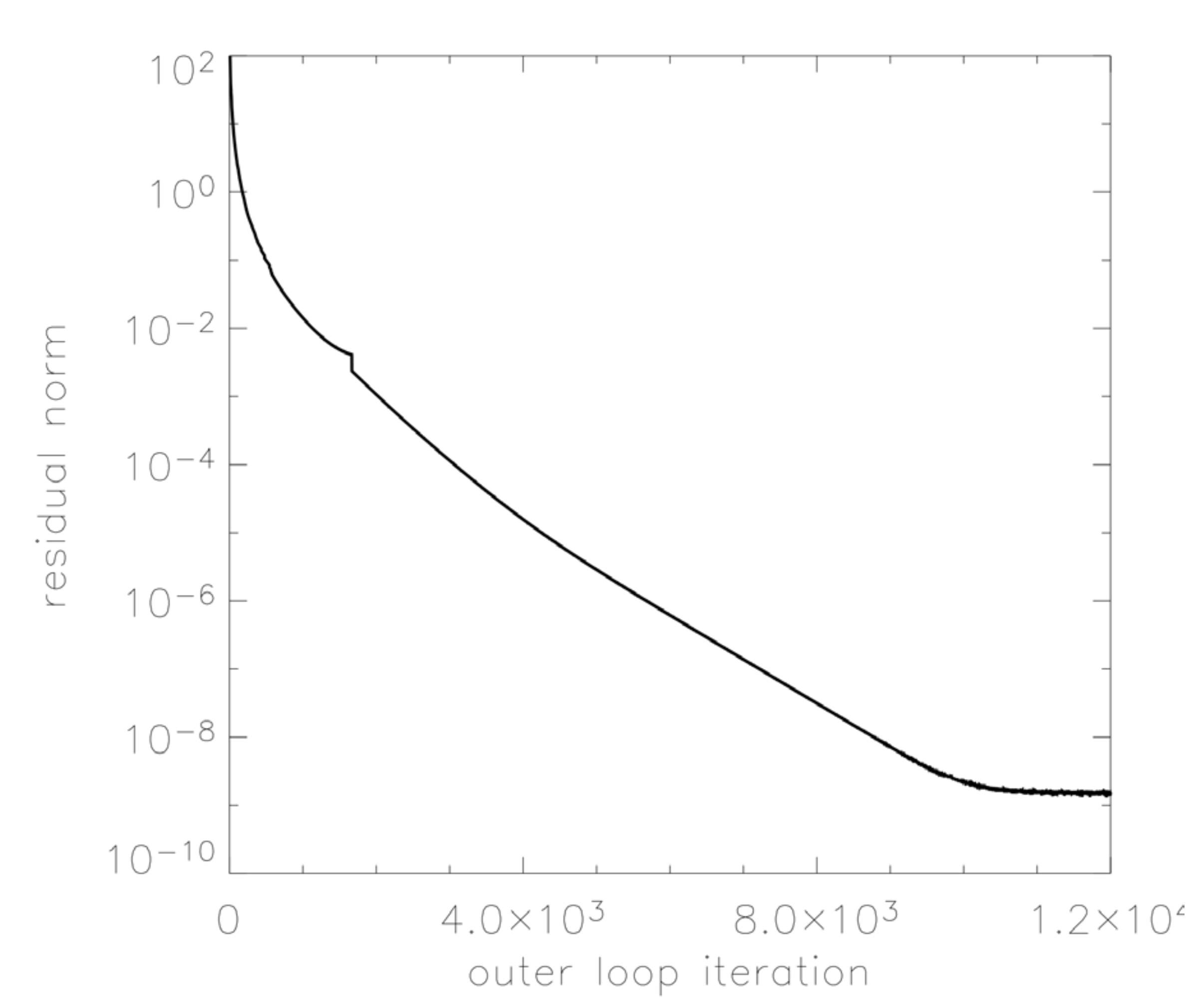
do $k=1, k_{max}$

Solve $A(u^{k-1})u^k=b$ with a linear solver

if $||F(u^k)|| < \gamma_{nl}||F(u^0)||$ stop

enddo

Numerical convergence of the nonlinear solution (6 January1997 00Z)



$\Delta t = 6\text{h}$

$\Delta x = 10\text{km}$

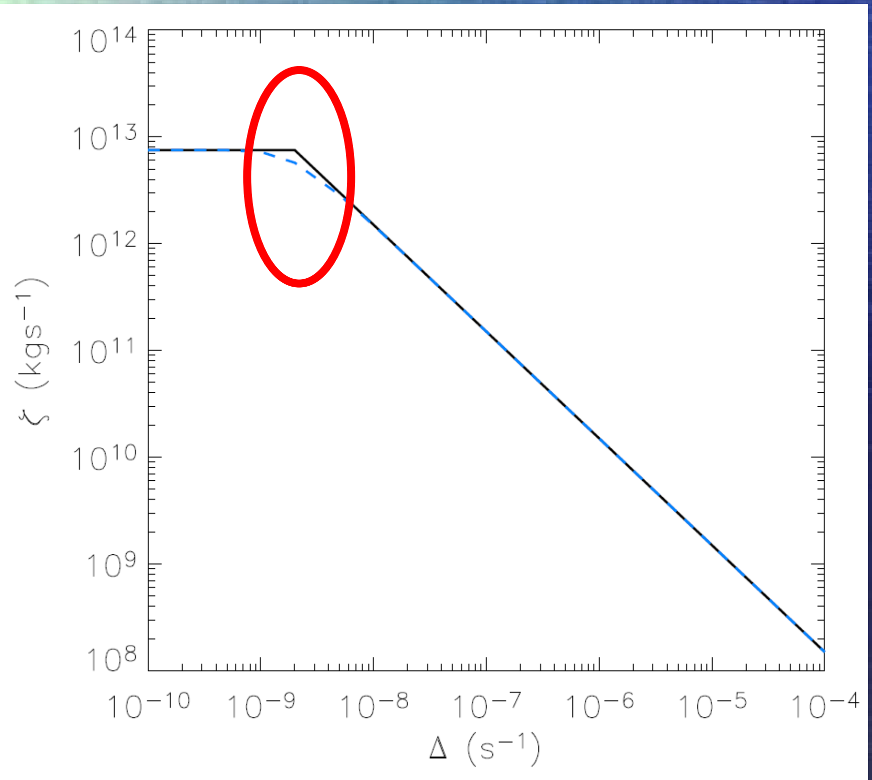
Why is the convergence so slow?

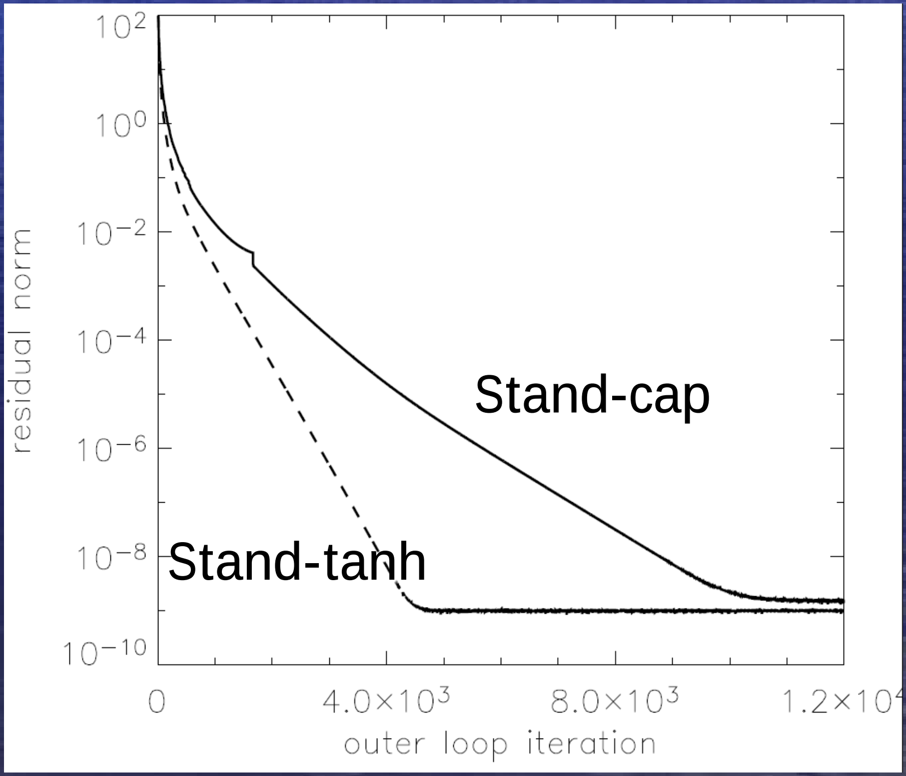
- The momentum equation with a VP formulation is highly nonlinear
- The equation is not continuously differentiable because of a capping of the viscous coefficients
- The linearization is not optimal

Continuously differentiable formulation

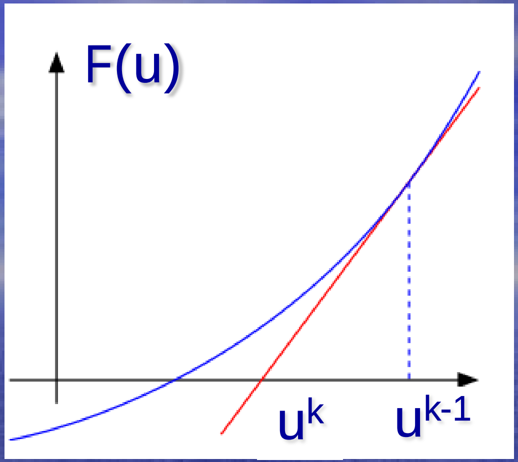
Discontinuous : $\zeta = \min(P/2\Delta, \zeta_{\max}), \;\; \eta = \zeta e^{-2}$

Continuous : $\zeta = \zeta_{\max} \tanh(P/2\Delta\zeta_{\max}), \;\; \eta = \zeta e^{-2}$





Standard linearization versus Newton



$F(u) = \zeta(u)u - b = 0$

Taylor series: $F(u^{k-1}+\delta u^k) = F(u^{k-1}) + F'\delta u^k$

Requiring $F(u^{k-1}+\delta u^k) = 0$, we get

$$\delta u^k = u^k - u^{k-1} = -F(u^{k-1})/F'$$

$$\text{where } F' = \zeta'(u^{k-1})u^{k-1} + \zeta(u^{k-1})$$

$\Rightarrow$ Standard / Newton: $u^k = u^{k-1} - F(u^{k-1}) \,/\, (\zeta'(u^{k-1})\, u^{k-1} + \zeta(u^{k-1}))$

The JFNK solver

At time t, we want to solve
the nonlinear system of equations:

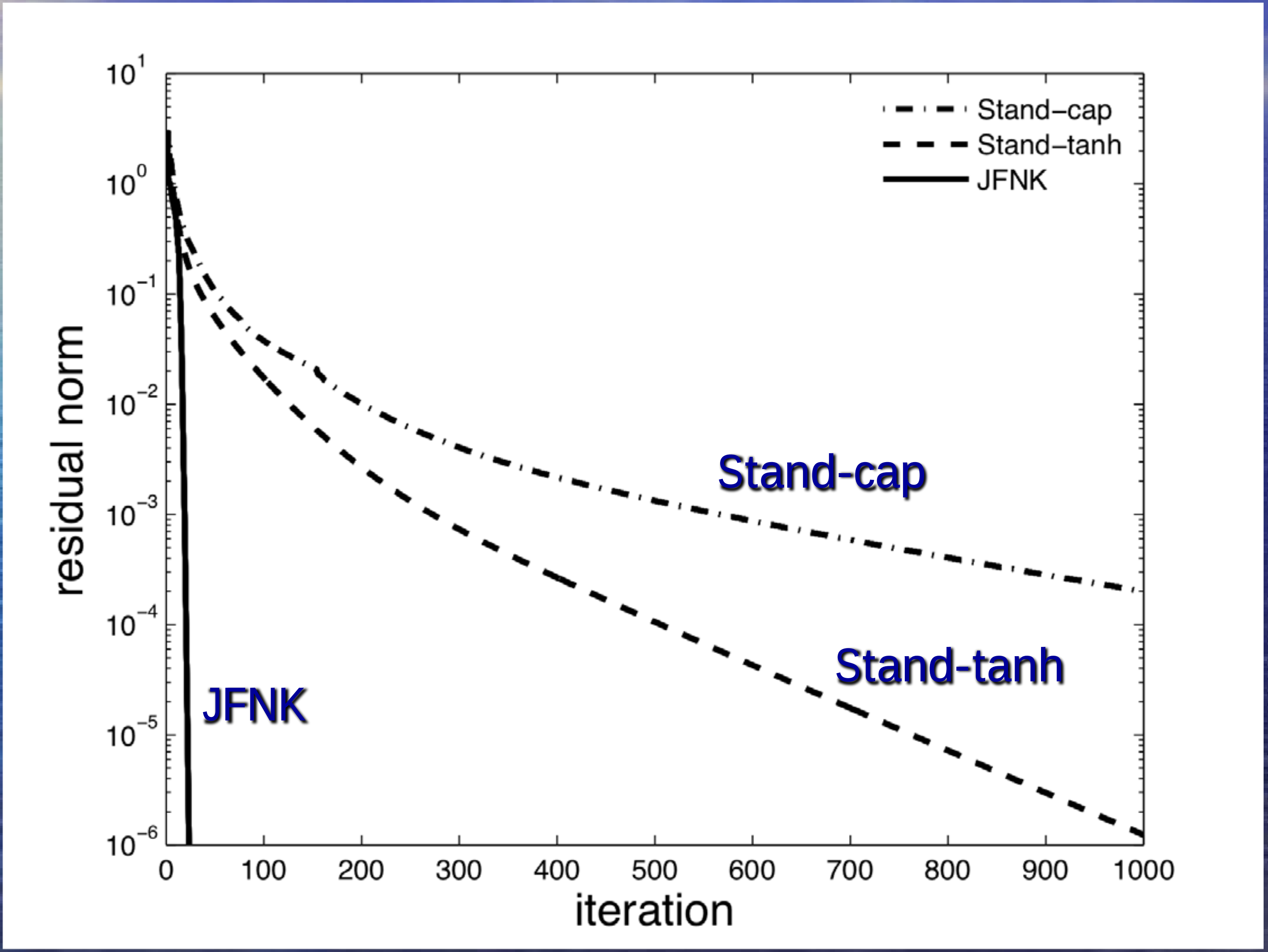
$$F(u)=A(u)u-b=0$$

u^0: initial iterate

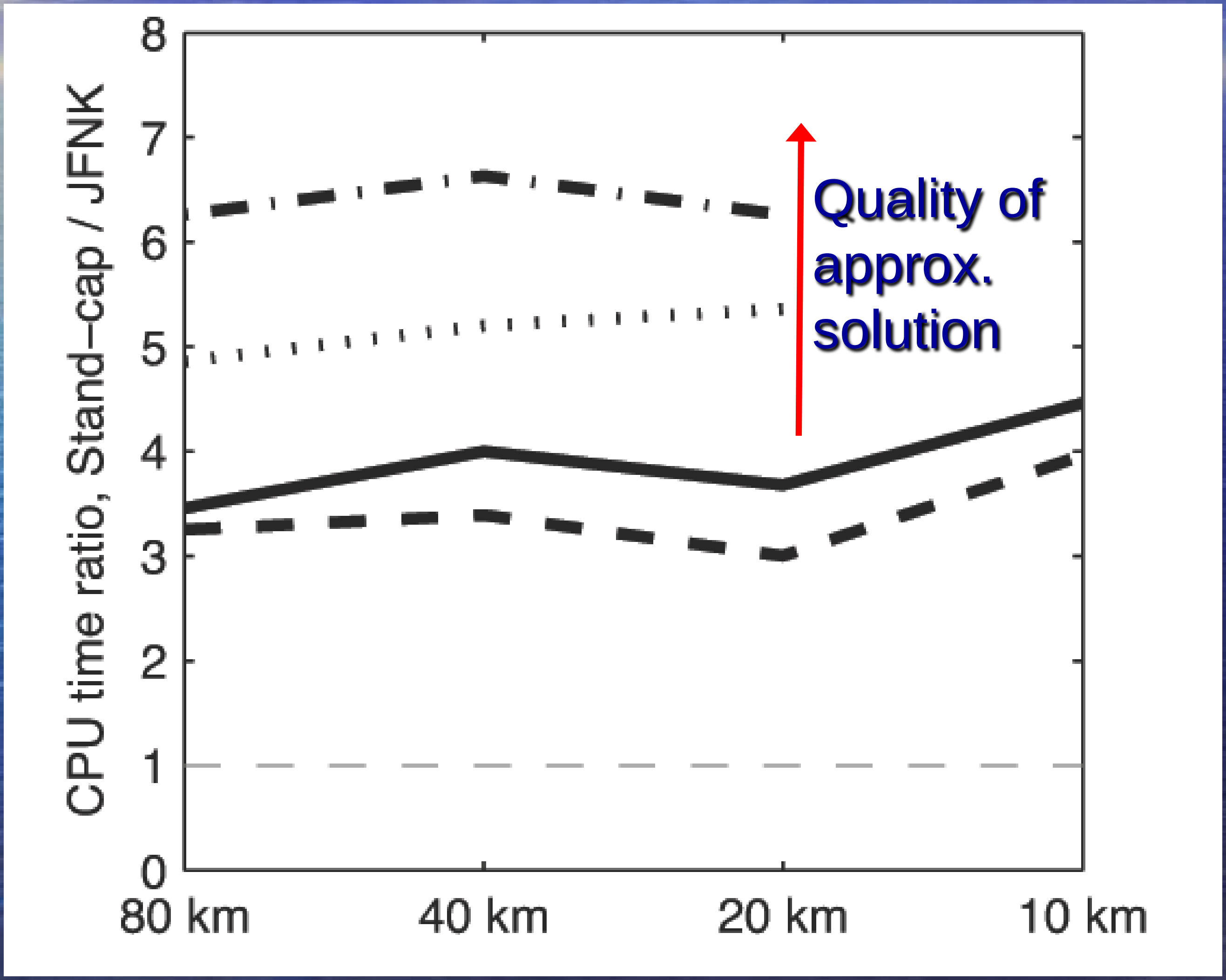
do $k=1, k_{max}$
"Solve" $J(u^{k-1})\delta u^k=-F(u^{k-1})$ with preconditioned GMRES
$u^k = u^{k-1} + \delta u^k$
if $||F(u^k)|| < \gamma_{nl}||F(u^0)||$ stop
enddo

$$J(u^{k-1})v \sim (F(u^{k-1}+\varepsilon v) - F(u^{k-1})) / \varepsilon$$

Comparison of the JFNK and standard solver



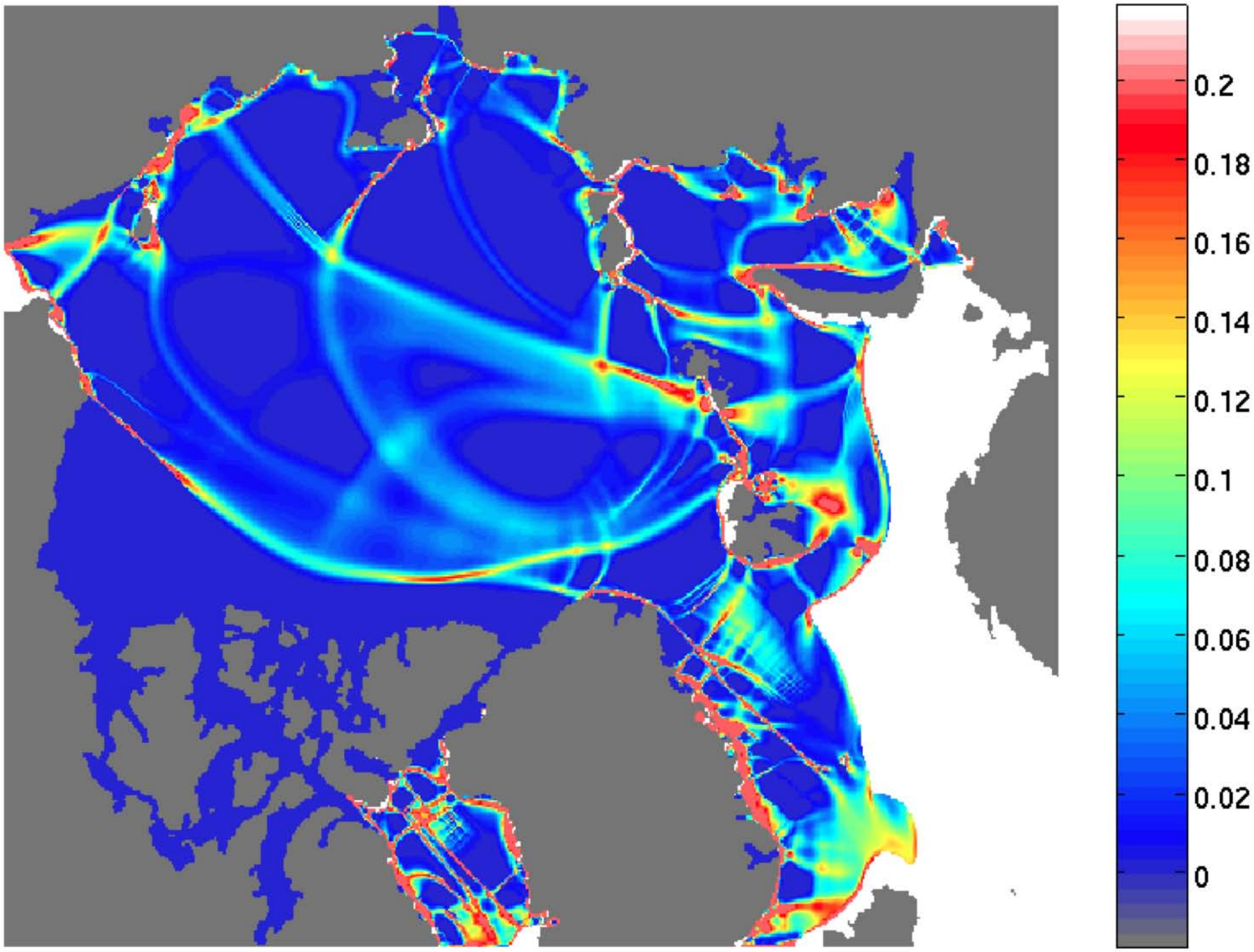
Computational gain of JFNK over the standard solver



Conclusions

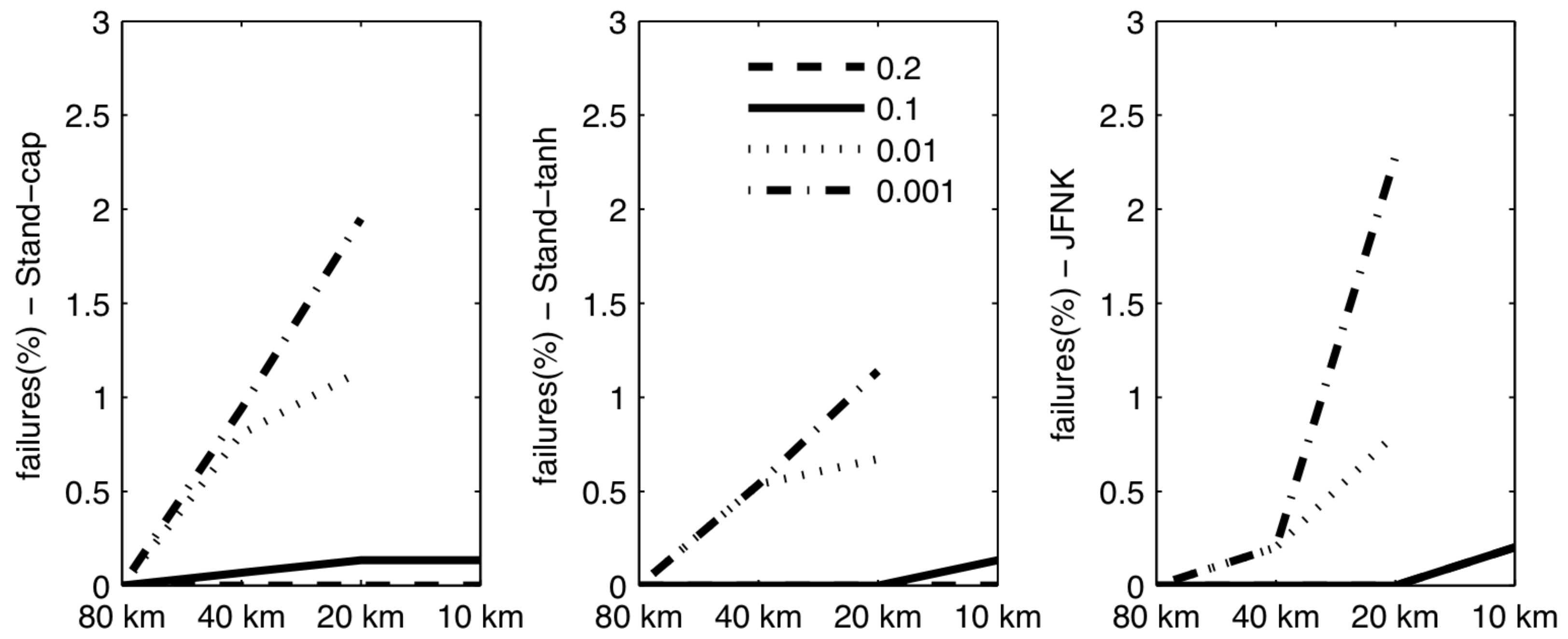
- The standard solver of VP models implies a very slow convergence
- The convergence can be improved by:
  - having a continuously differentiable equation
  - using a Newton method
- Both standard and JFNK solvers show a certain lack of robustness

Typical shear deformation field (10 km)

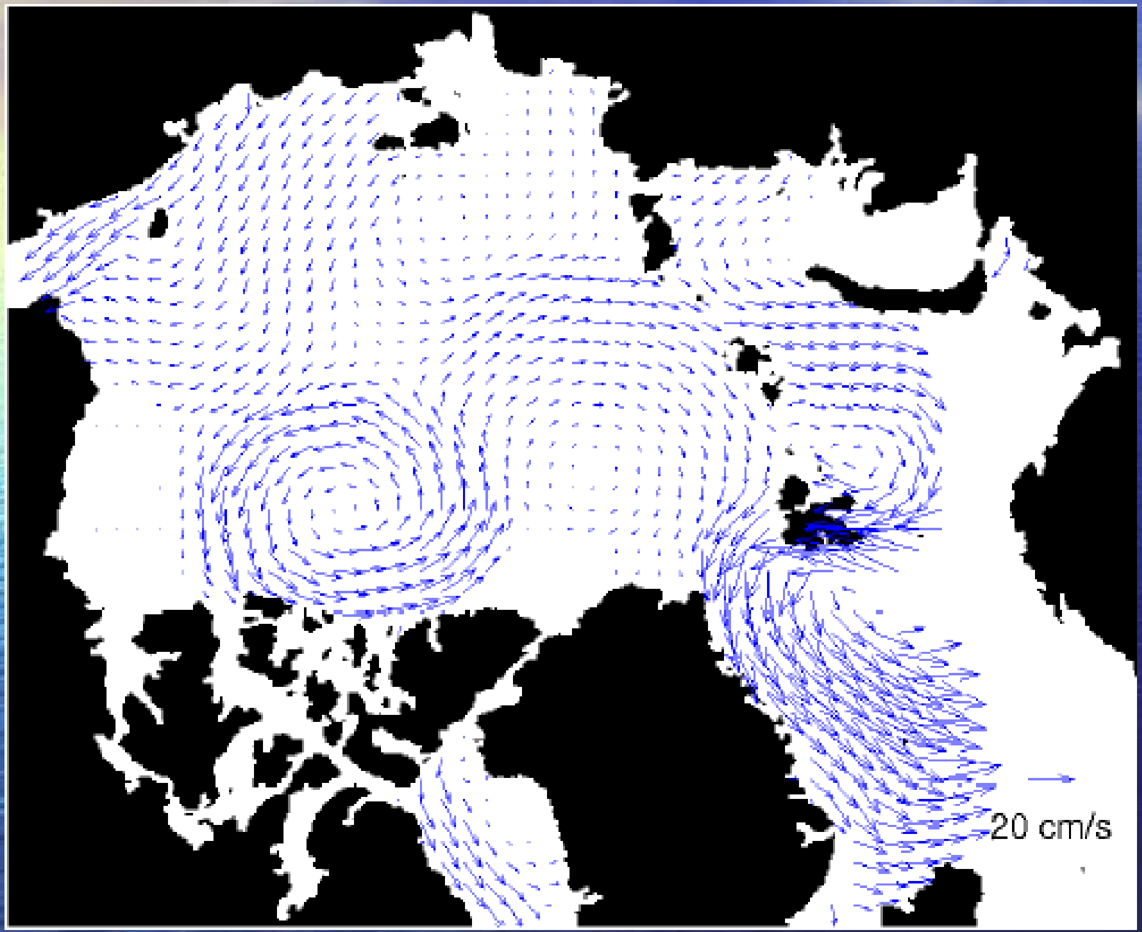


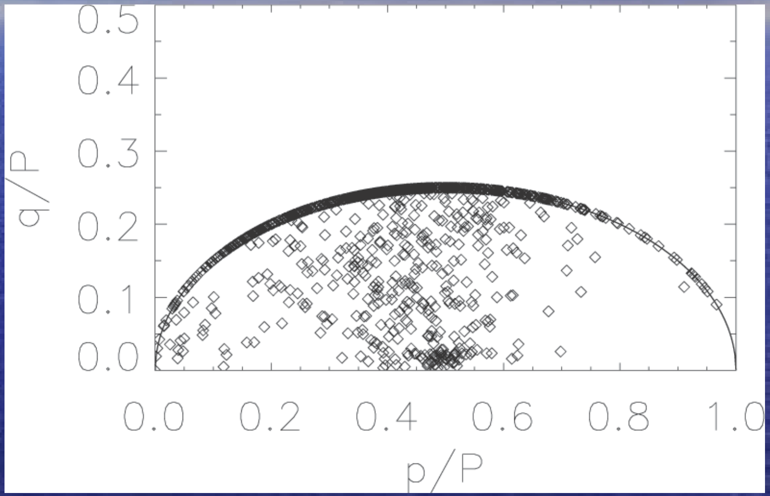
Thank you!

Failures of the two solvers



The fully converged solution
6 January1997 00Z





$\Delta t = 6h$

$\Delta x = 10km$

In matrix form...

$$F(u)=A(u)u-b=0$$

Standard method (iteration k)

$$A(u^{k-1})u^k = b$$

$$u^k = u^{k-1} - A^{-1}(u^{k-1})F(u^{k-1})$$

$$\mathbf{u}^k = \mathbf{u}^{k-1} - \mathbf{A}^{-1}(\mathbf{u}^{k-1})\mathbf{F}(\mathbf{u}^{k-1}).$$

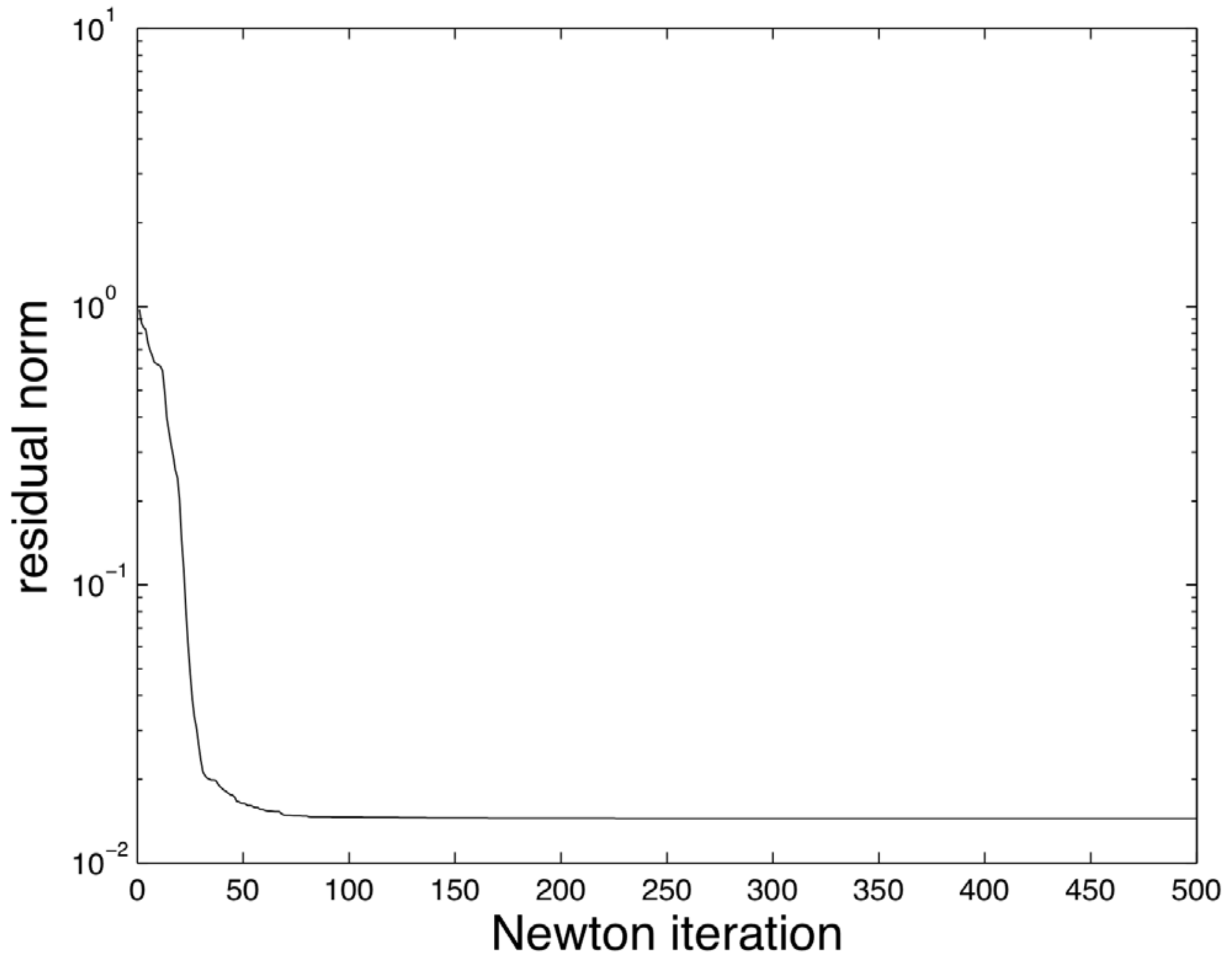
Newton method (iteration k)

$$J(u^{k-1})\delta u^k = -F(u^{k-1})$$

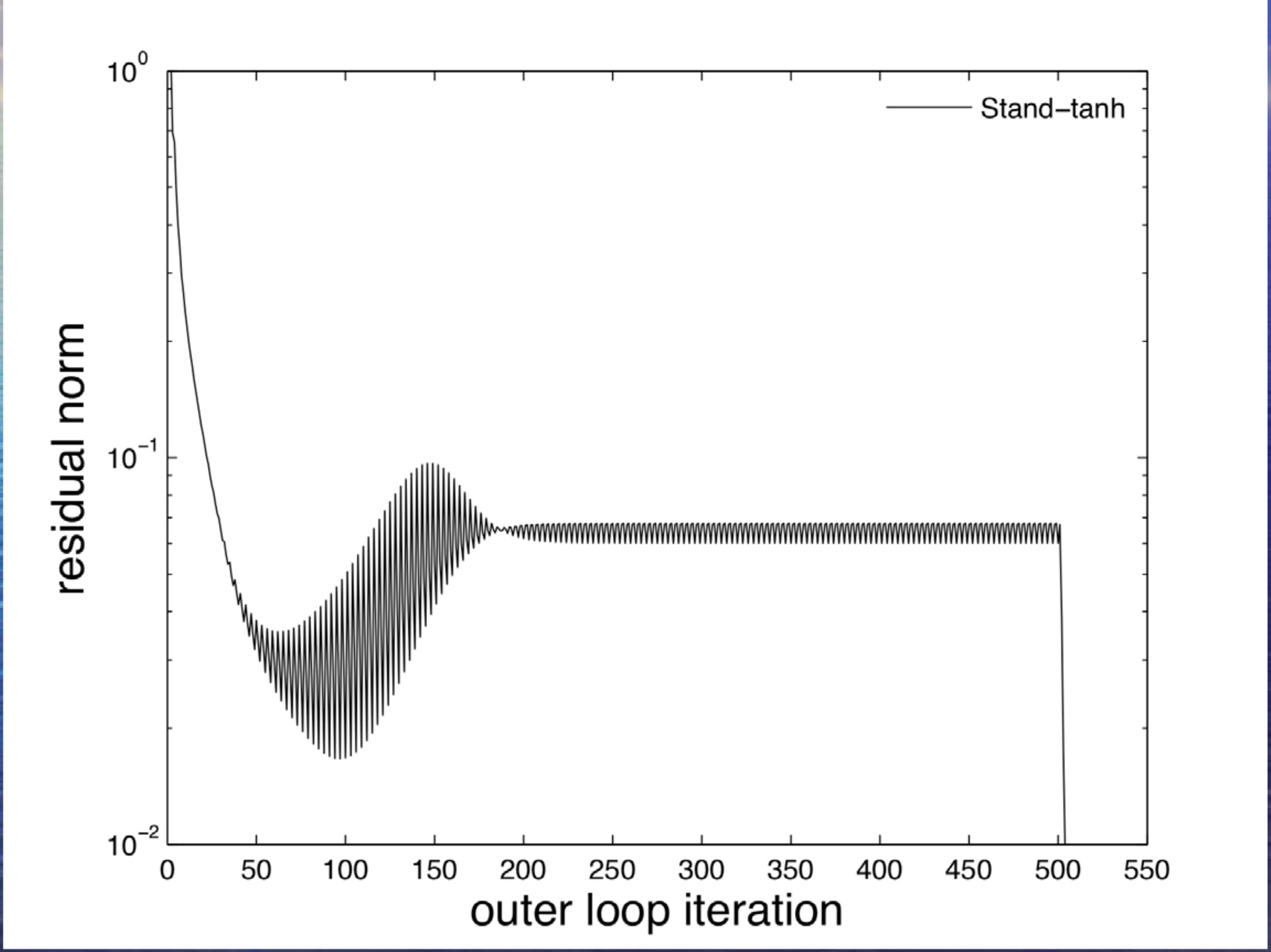
$$u^k = u^{k-1} - J^{-1}(u^{k-1})F(u^{k-1})$$

$$u^k = u^{k-1} - (A(u^{k-1})+G(u^{k-1}))^{-1}F(u^{k-1})$$

Failure of the line search method



Globalization method for the standard solver



Coming soon...

- méthode de convergence globale (Paul Tupper)
- préconditionneur multi-grille algébrique (Paul Tupper)
- parallélisation (Andy Pintar)

Dynamique de la glace de mer

$$\rho_i h \frac{Du}{Dt} = -\rho_i h f \hat{k} \times u + \tau_a - \tau_w - \rho_i h g \nabla H_d + \nabla \cdot \sigma$$

terme de rhéologie

$$\tau_a = \rho_a C_{da} |u_a^g| \left(u_a^g \cos\theta_a + k \times u_a^g \sin\theta_a \right)$$

$$\tau_w = \rho_w C_{dw} |u - u_w^g| \left[(u - u_w^g) \cos\theta_w + k \times (u - u_w^g) \sin\theta_w \right]$$

Schéma numérique explicite?

$$\rho_i h \frac{Du}{Dt} = -\rho_i h f \hat{k} \times u + \tau_a - \tau_w - \rho_i h g \nabla H_d + \nabla \cdot \sigma$$

Analyse de stabilité

$$\rho_i h \frac{\partial u}{\partial t} = \zeta_{\max} \frac{\partial^2 u}{\partial x^2} + R \Rightarrow \Delta t \leq \frac{\rho_i h}{2\zeta_{\max}} \Delta x^2$$

Schéma numérique implicite

Nous allons résoudre de façon implicite au temps:

...t-Δt, t, t+Δt, t+2Δt, ...

Au temps t nous avons:

$$-\frac{\rho_i h u^t}{\Delta t} + \rho_i h f v^t_{avg} + \frac{\partial}{\partial x}\left[(\eta+\zeta)\frac{\partial u^t}{\partial x}\right] + \frac{\partial}{\partial x}\left[(\zeta-\eta)\frac{\partial v^t}{\partial y}\right] +$$

$$\frac{\partial}{\partial y}\left[\eta\frac{\partial u^t}{\partial y}\right] + \frac{\partial}{\partial y}\left[\eta\frac{\partial v^t}{\partial x}\right] - C_w\left(u^t\cos\theta_w - v^t_{avg}\sin\theta_w\right) = b_u$$

Schéma numérique implicite

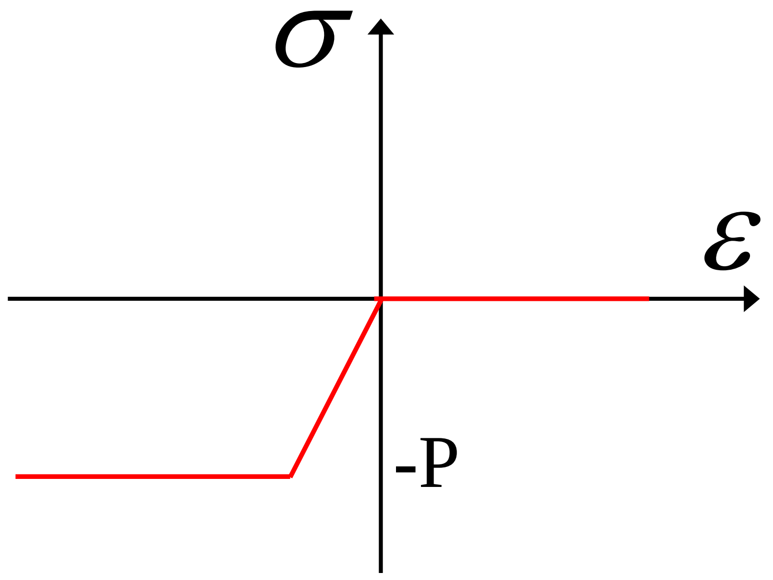
Nous laissons tomber l'indice t.

À l'itération k nous avons:

$$-\frac{\rho_i h u^k}{\Delta t} + \rho_i h f v^k_{avg} + \frac{\partial}{\partial x}\left[\left(\eta(u_l) + \zeta(u_l)\right)\frac{\partial u^k}{\partial x}\right] + \frac{\partial}{\partial x}\left[\left(\zeta(u_l) - \eta(u_l)\right)\frac{\partial v^k}{\partial y}\right] +$$

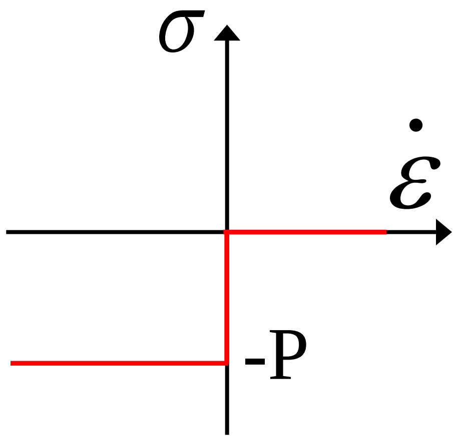
$$\frac{\partial}{\partial y}\left[\eta(u_l)\frac{\partial u^k}{\partial y}\right] + \frac{\partial}{\partial y}\left[\eta(u_l)\frac{\partial v^k}{\partial x}\right] - C_w(u_l)\left(u^k \cos\theta_w - v^k_{avg} \sin\theta_w\right) = b_u$$

Rhéologie de la glace de mer en 1D



elasto-plastique

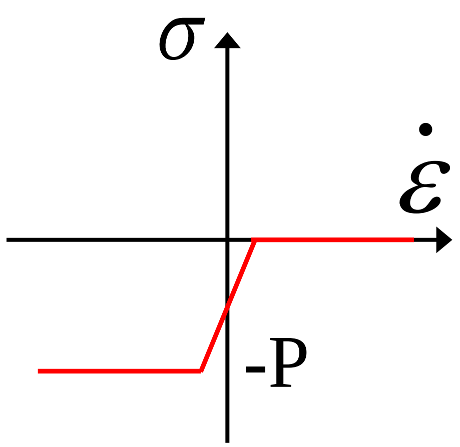
Rhéologie de la glace de mer en 1D



plastique

$$\sigma = \zeta \dot{\varepsilon} - \frac{P}{2}$$

$$\zeta = \frac{P}{2|\dot{\varepsilon}|}$$



visco-plastique

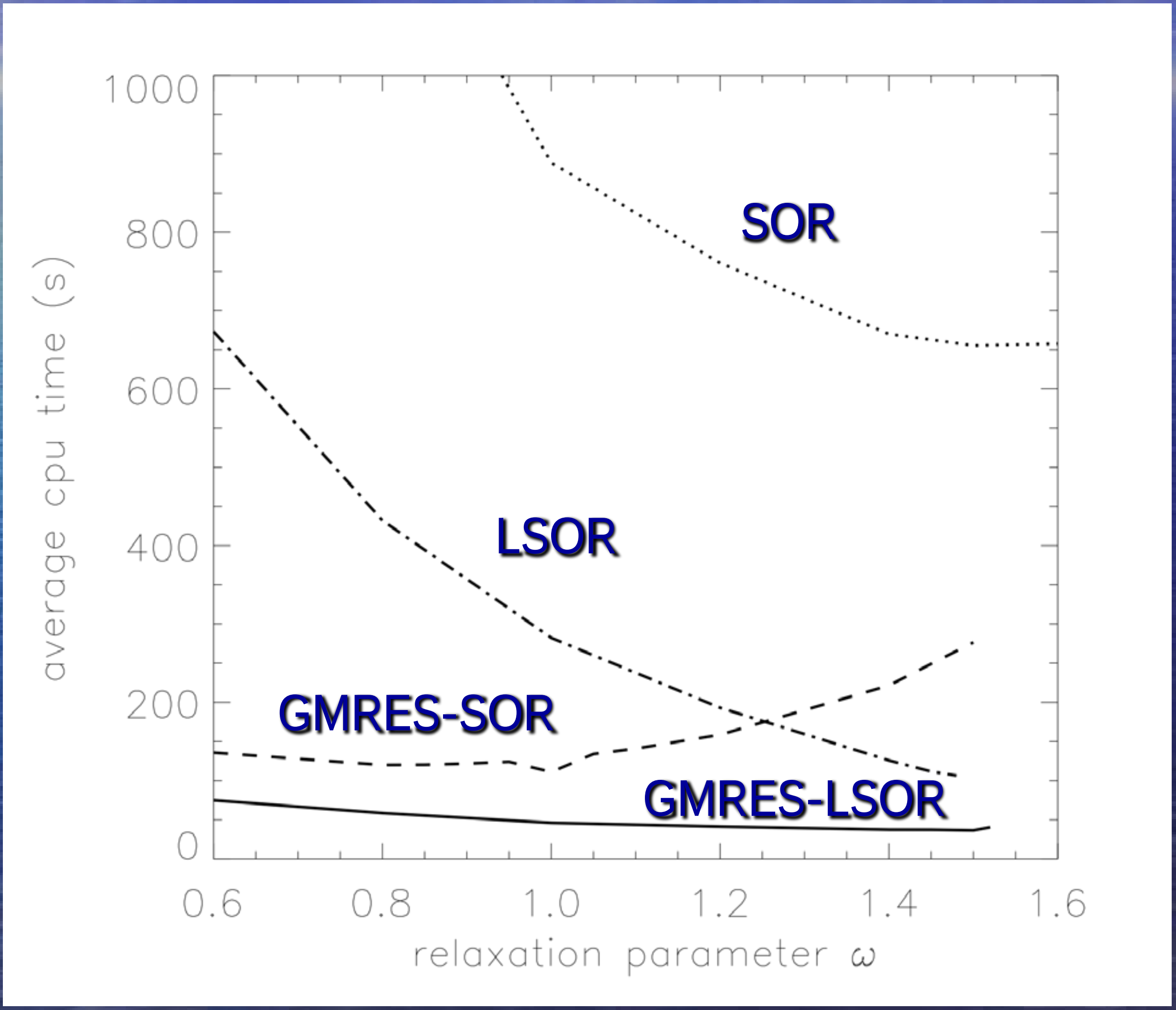
$$\sigma = \zeta \dot{\varepsilon} - \frac{P}{2}$$

$$\zeta = \min\left(\frac{P}{2|\dot{\varepsilon}|}, \zeta_{\max}\right)$$

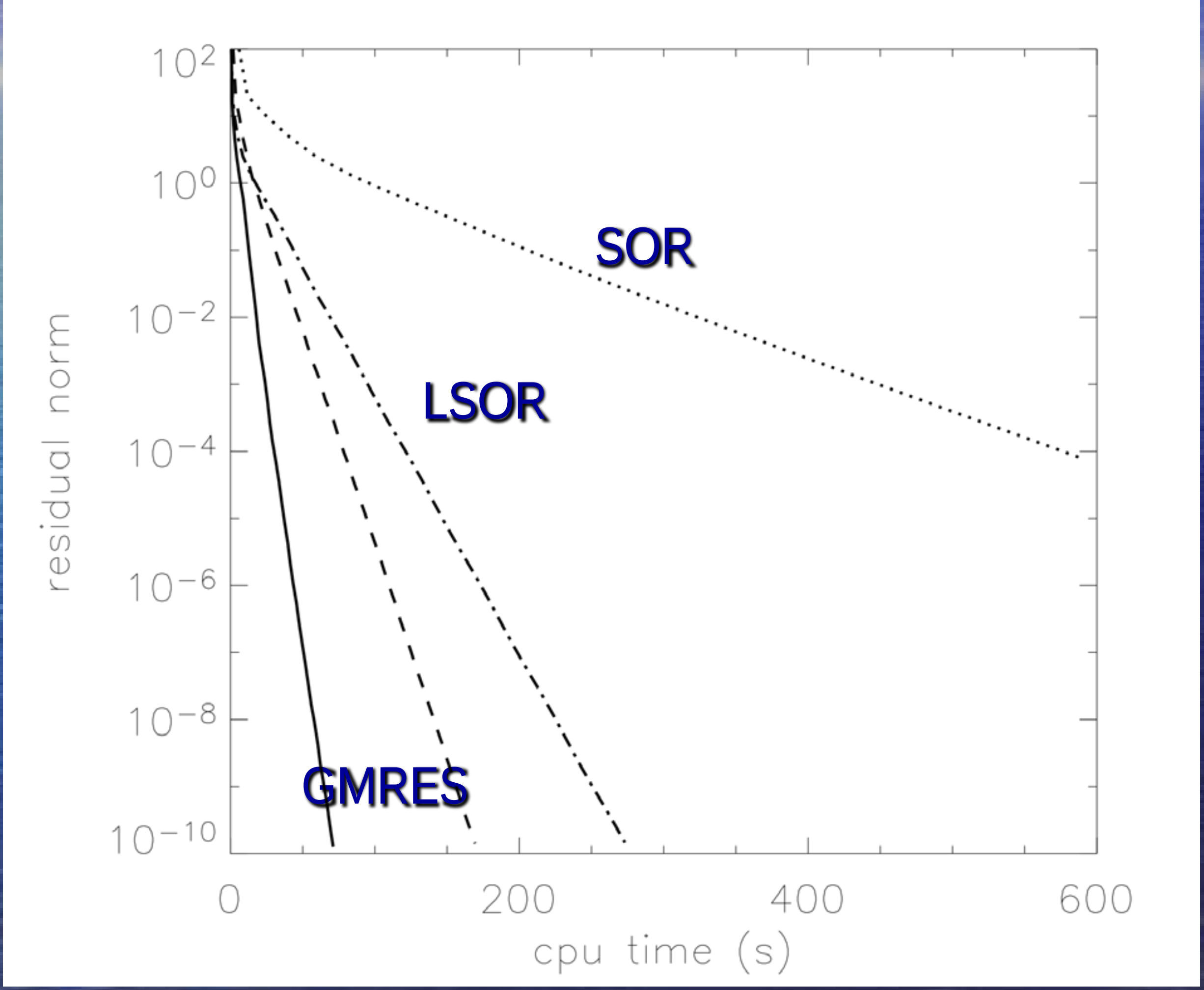
Méthode du Résidu Minimal Généralisé (GMRES) avec préconditionneur

- méthode de sous-espaces de Krylov
- nécessite peu de mémoire
- la symétrie n’est pas nécessaire
- bonne convergence avec préconditionneur
- parallélisable

Optimisation et comparaison des solveurs linéaires

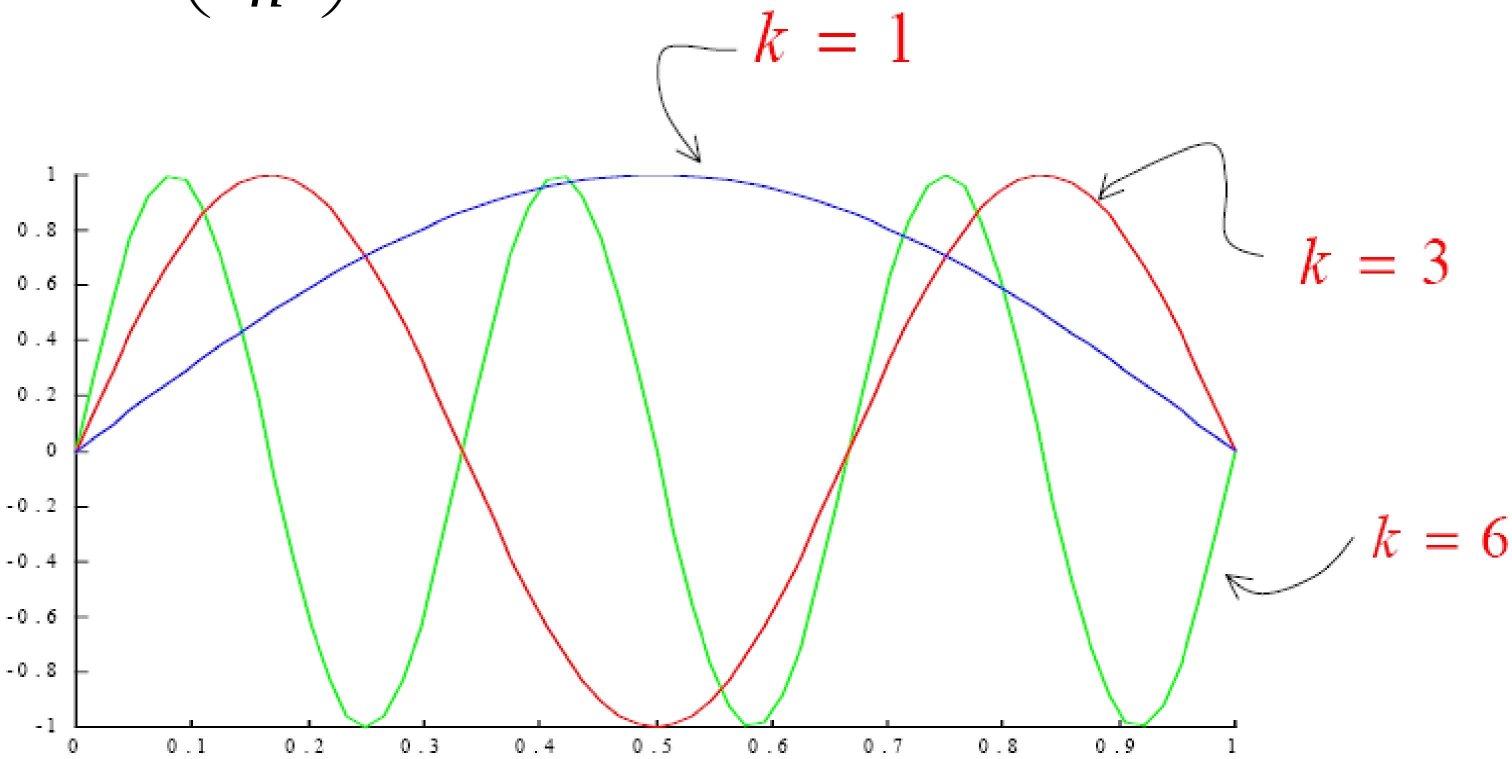


Taux de convergence des solveurs linéaires



Méthodes stationnaires:
Jacobi, Gauss-Seidel, SOR et LSOR

$$\sin\left(\frac{ik\pi}{n}\right) \qquad 1 \le i \le n-1, \quad 1 \le k \le n-1$$

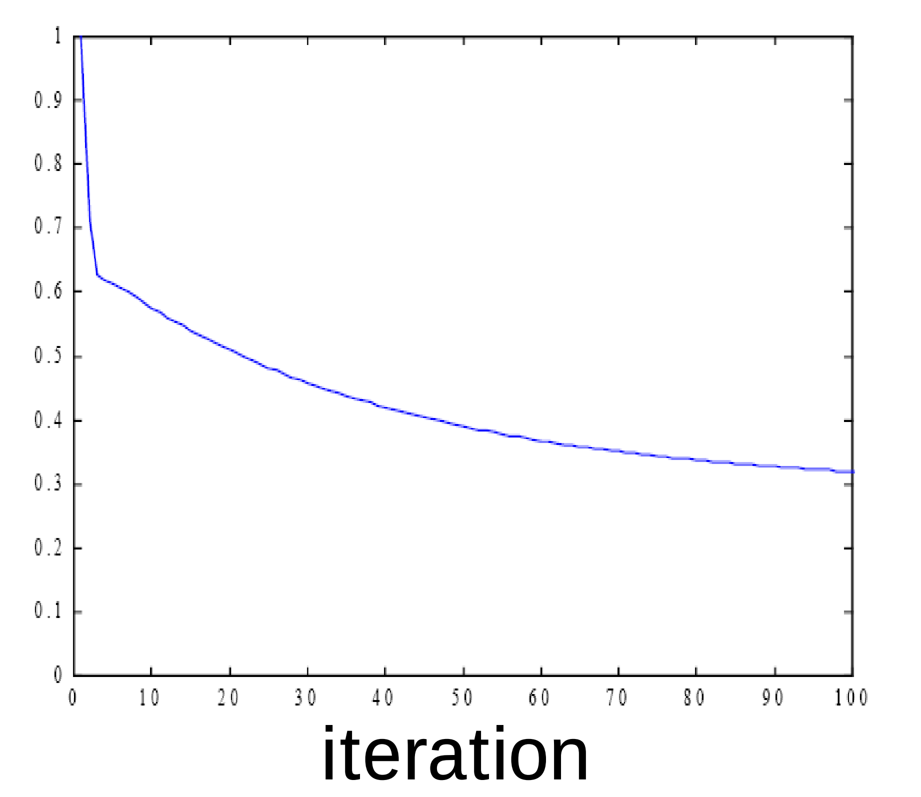


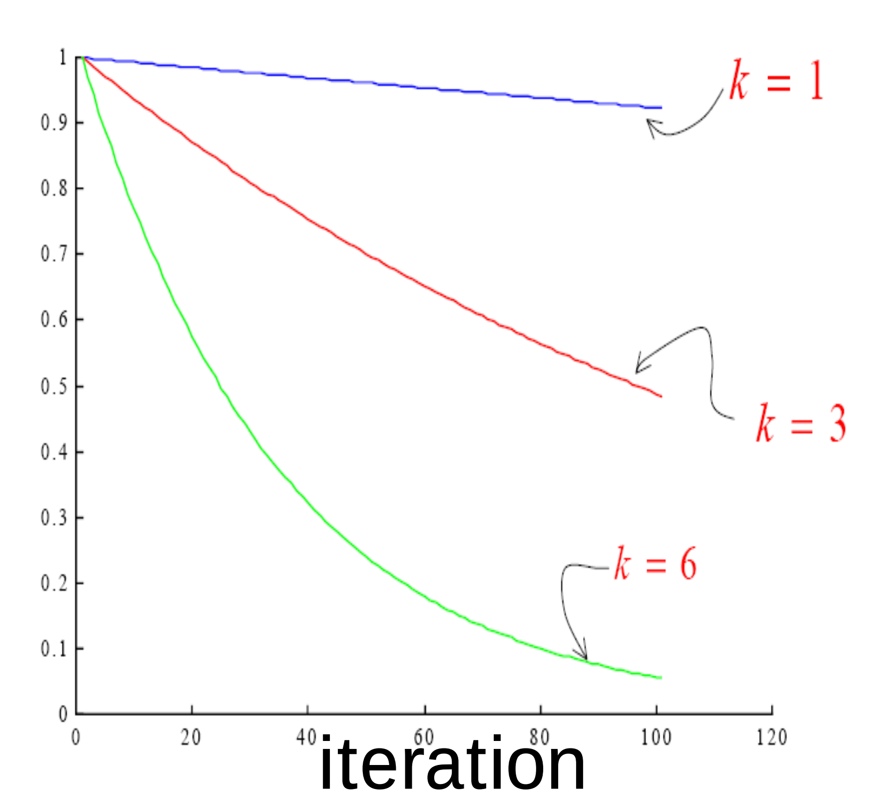
W. L. Briggs, a multigrid tutorial

Méthodes stationnaires: un exemple avec la méthode Jacobi

$$e_0 = \frac{1}{3}\left[\sin\left(\frac{i\pi}{n}\right) + \sin\left(\frac{3i\pi}{n}\right) + \sin\left(\frac{6i\pi}{n}\right)\right]$$

$$e_0 = \sin\left(\frac{ik\pi}{n}\right)$$, k=1 ou 3 ou 6





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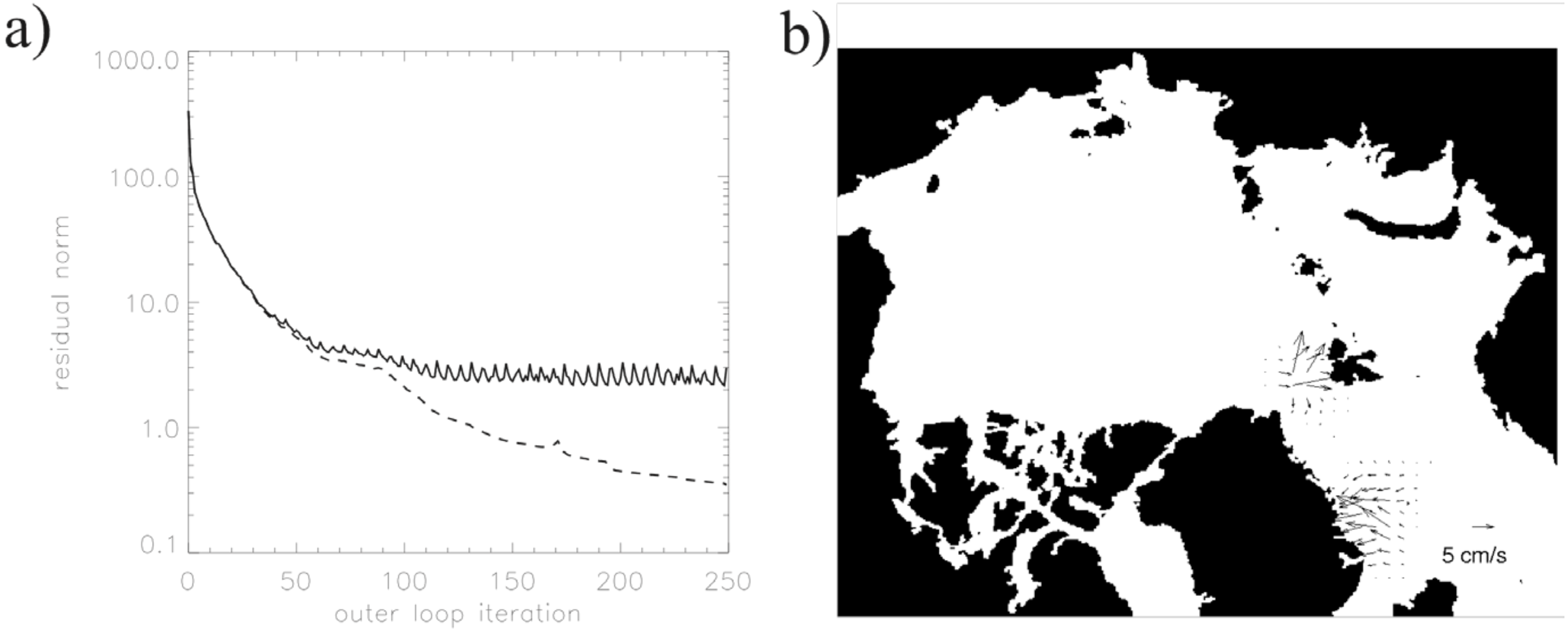
Traitement implicite du terme de Coriolis



$$\left|u_l^k - u_w^g\right| > \frac{\rho_i h f}{\rho_w C_{dw}(\cos\theta_w - \sin\theta_w)}$$

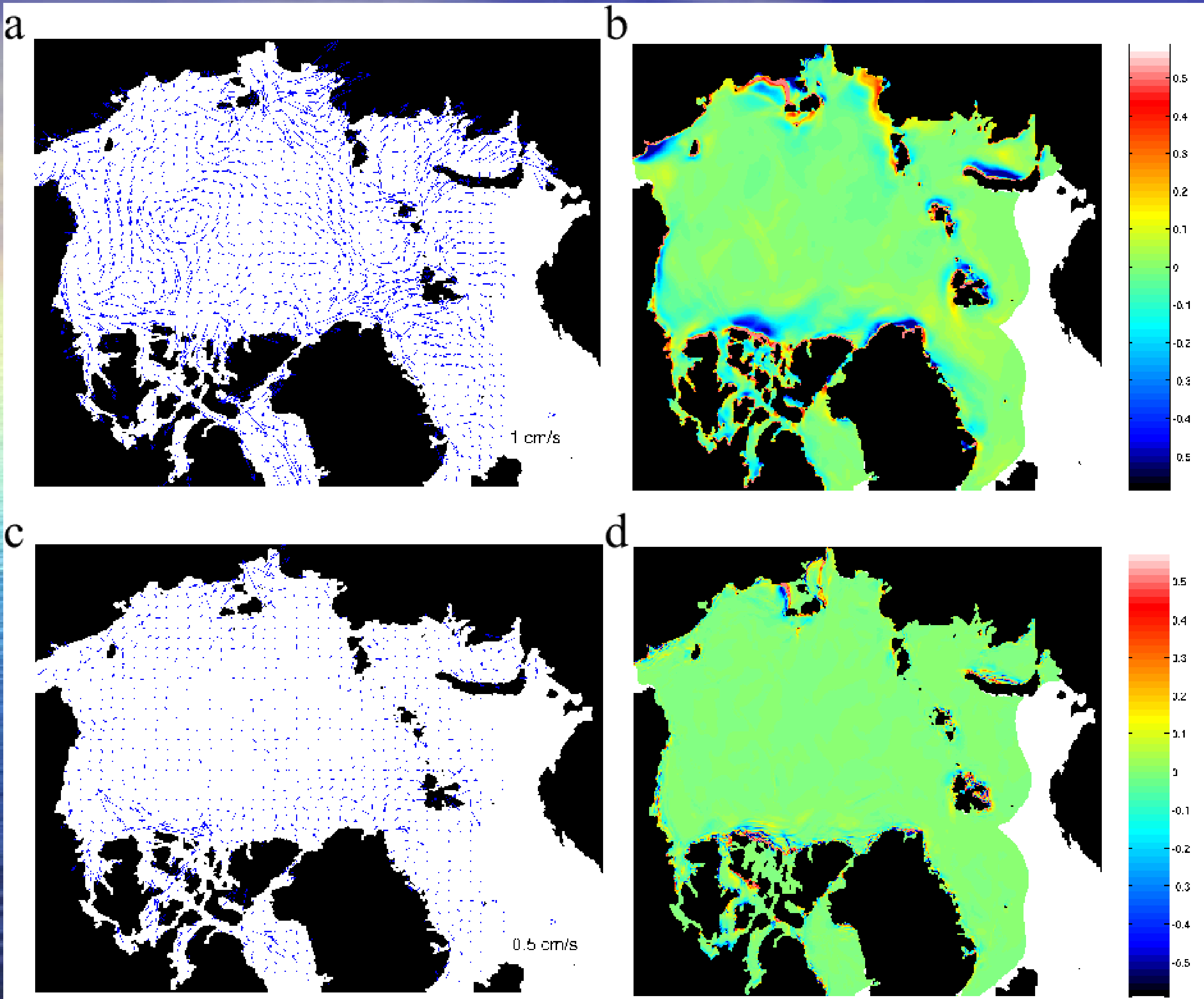
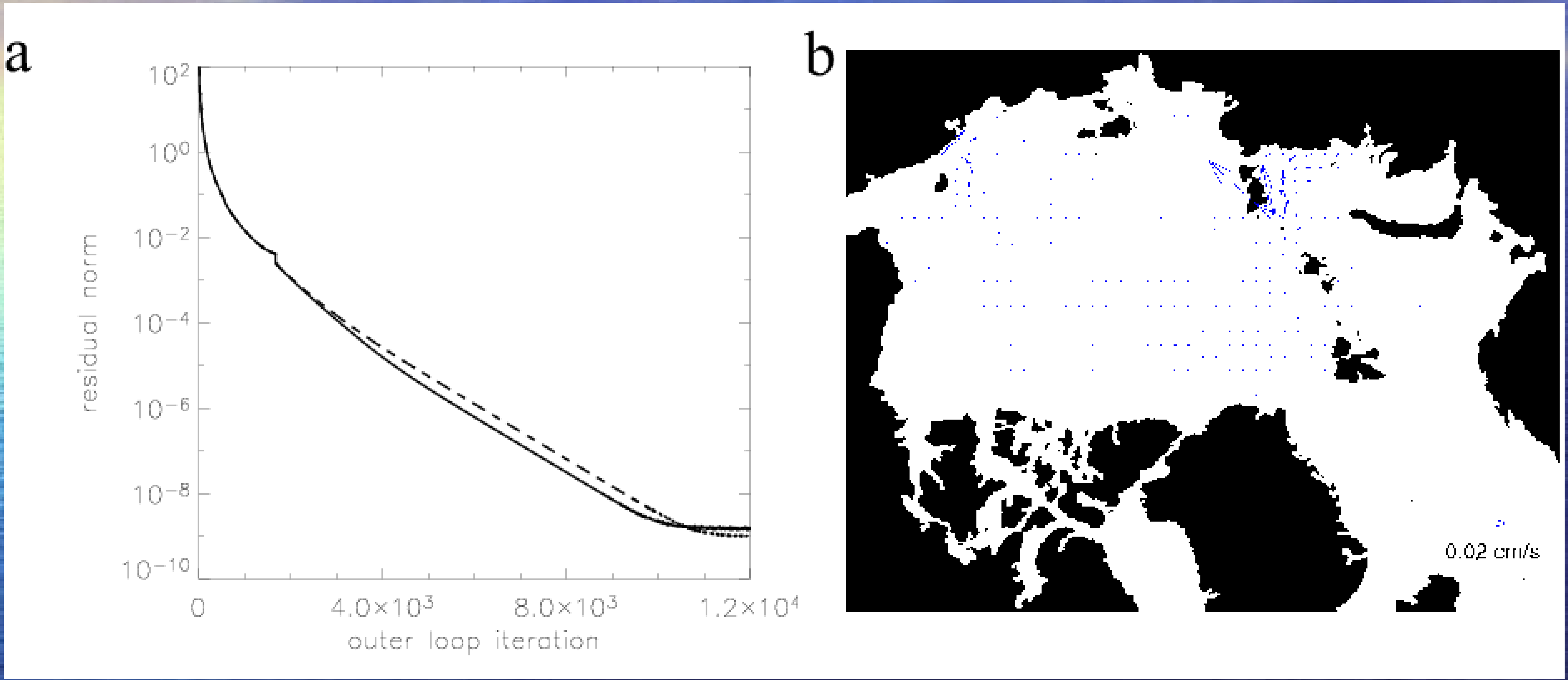


Fig 3-7

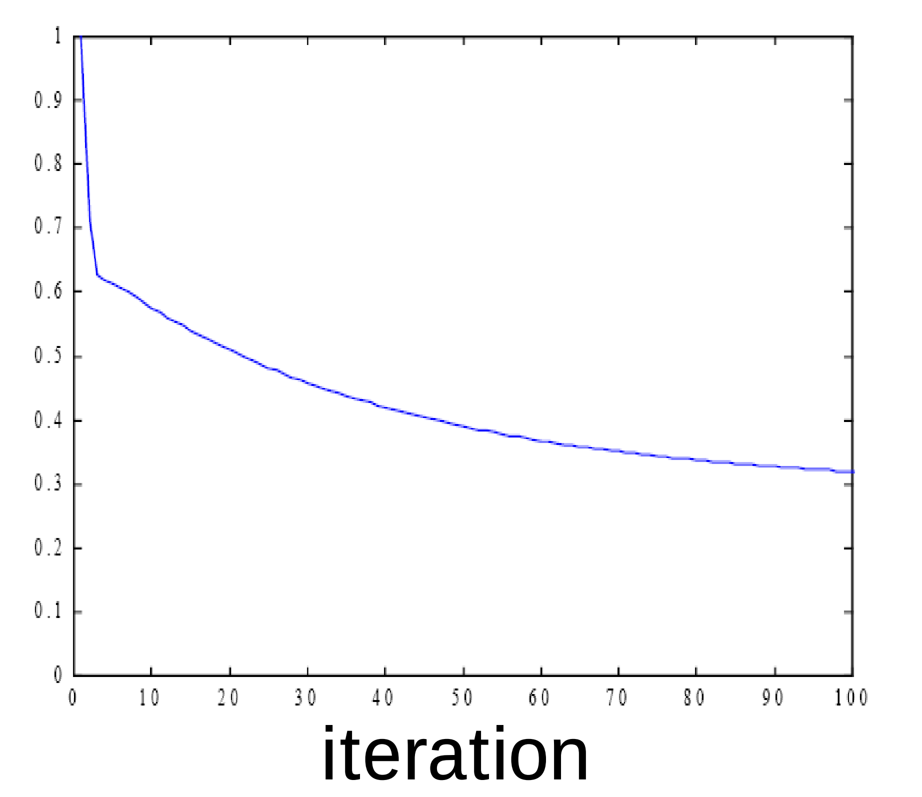
Fig 3-11

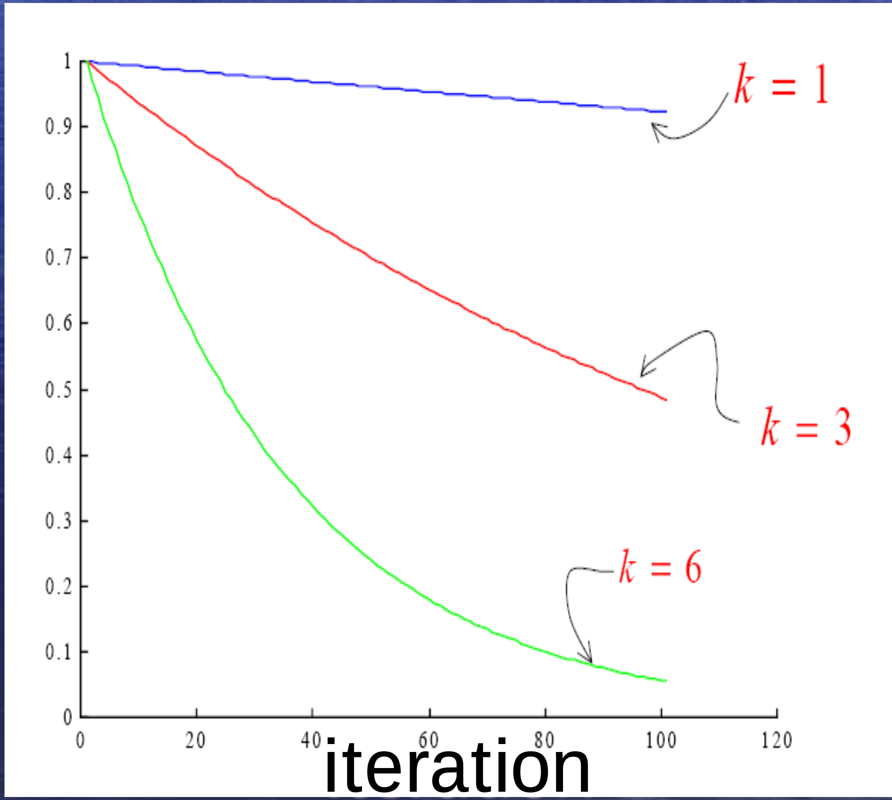

a
10^2
10^0
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
residual norm
0
4.0×10^3
8.0×10^3
1.2×10^4
outer loop iteration
b
0.02 cm/s

Méthodes stationnaires: un exemple avec la méthode Jacobi

$$e_0 = \frac{1}{3}\left[\sin\left(\frac{i\pi}{n}\right) + \sin\left(\frac{3i\pi}{n}\right) + \sin\left(\frac{6i\pi}{n}\right)\right]$$

$$e_0 = \sin\left(\frac{ik\pi}{n}\right)$$,k=1 ou 3 ou 6





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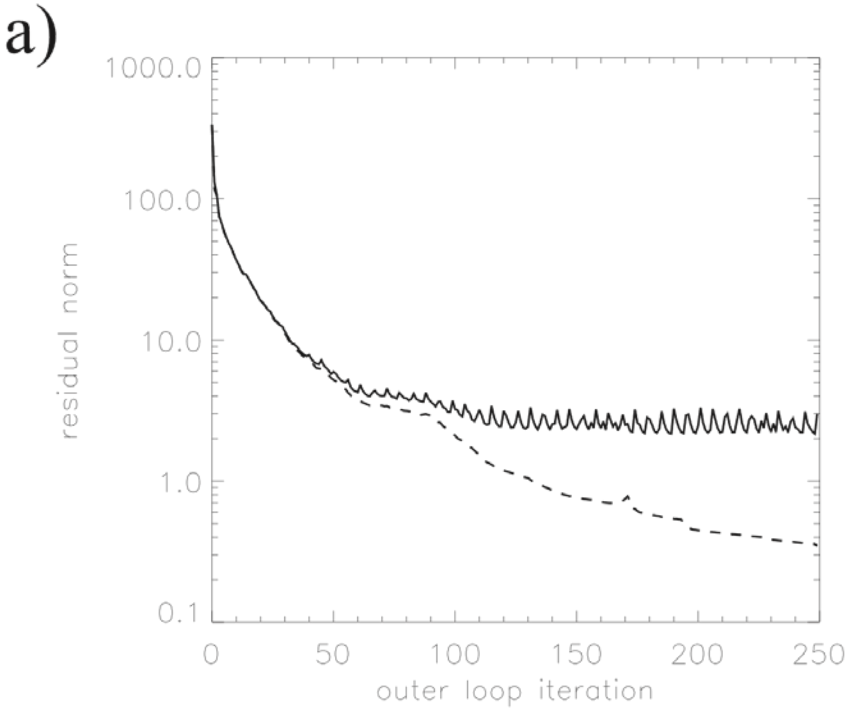
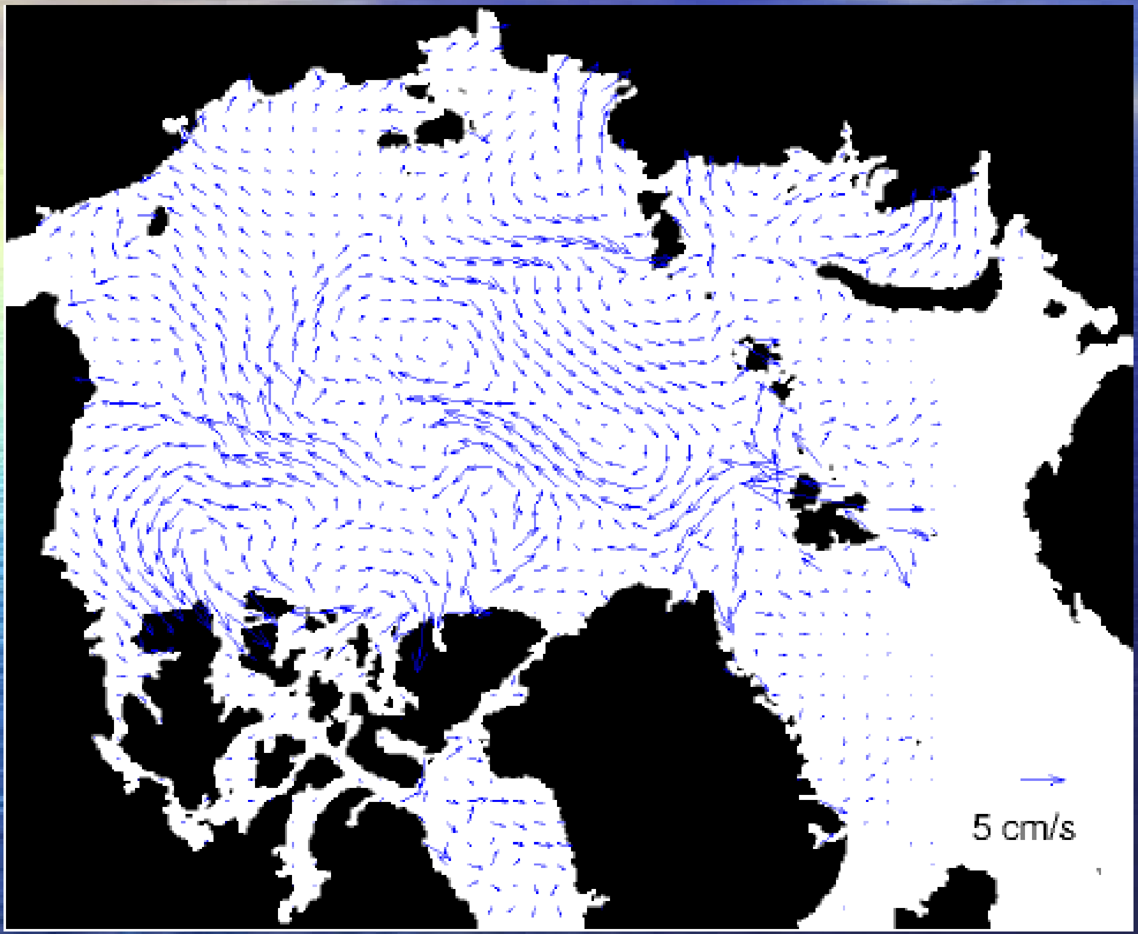
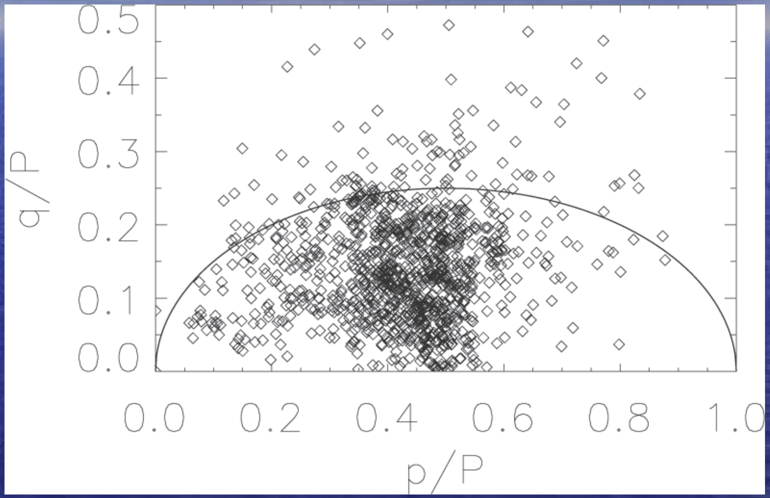
Traitement implicite du terme de Coriolis



$$\left| u_l^k - u_w^g \right| > \frac{\rho_i h f}{\rho_w C_{dw} (\cos\theta_w - \sin\theta_w)}$$

Erreurs après 2 itérations
6 janvier 1997 00Z

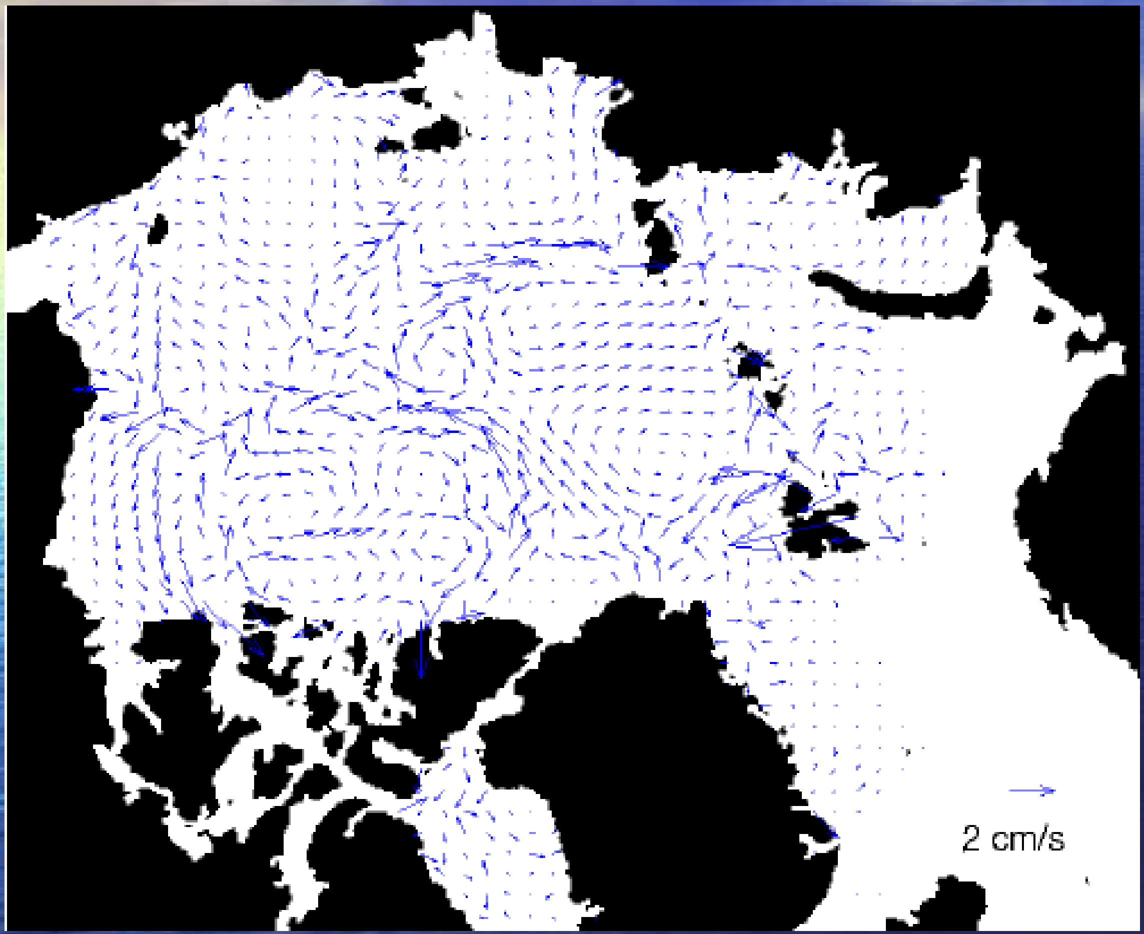


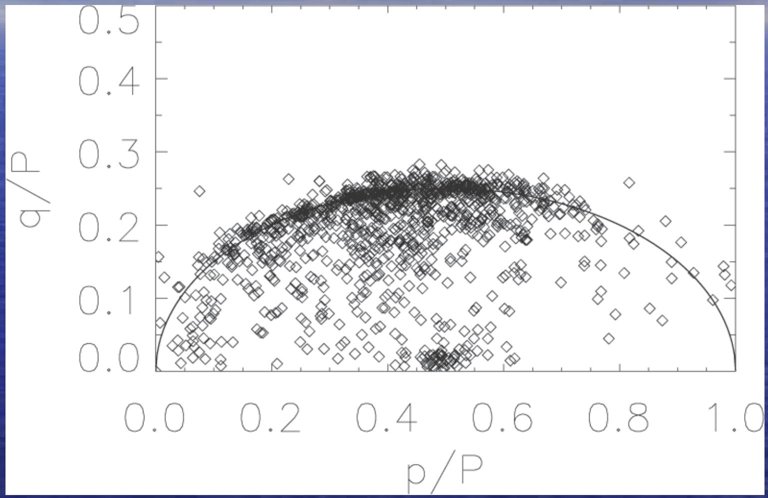


$\Delta t = 6h$

$\Delta x = 10km$

Erreurs après 10 itérations
6 janvier 1997 00Z

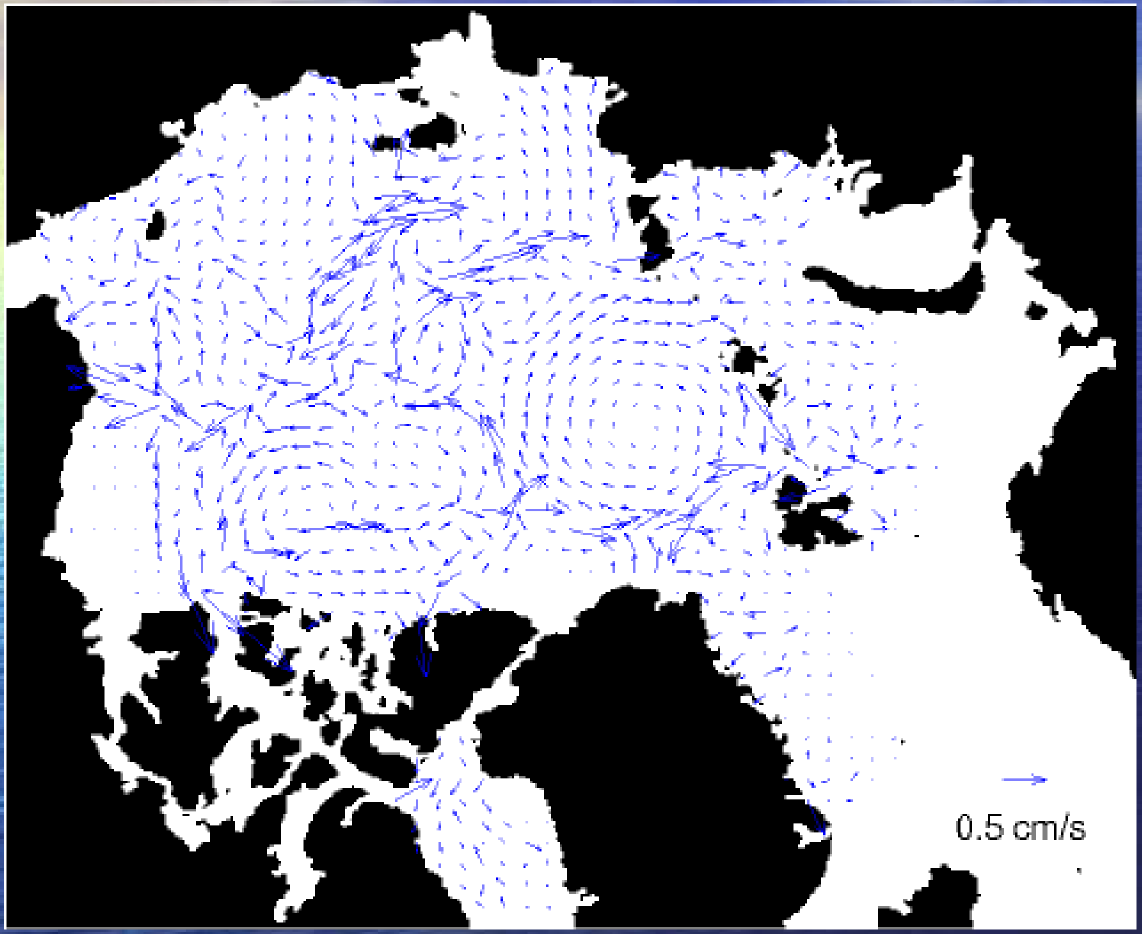


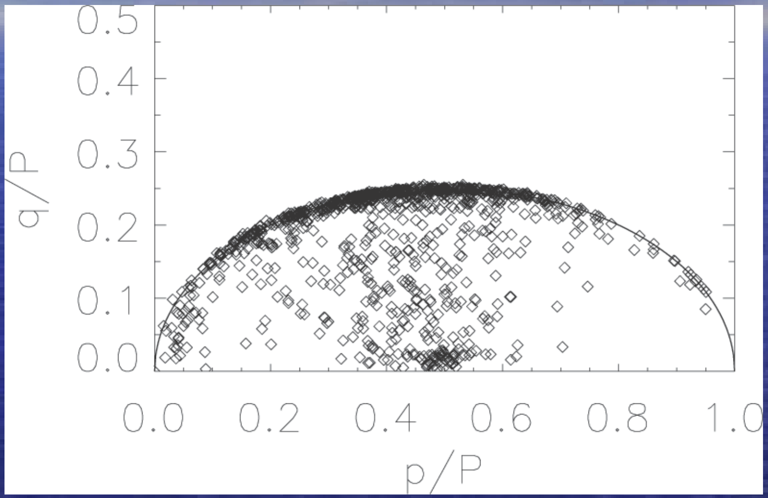


$\Delta t = 6h$

$\Delta x = 10km$

Erreurs après 40 itérations
6 janvier 1997 00Z



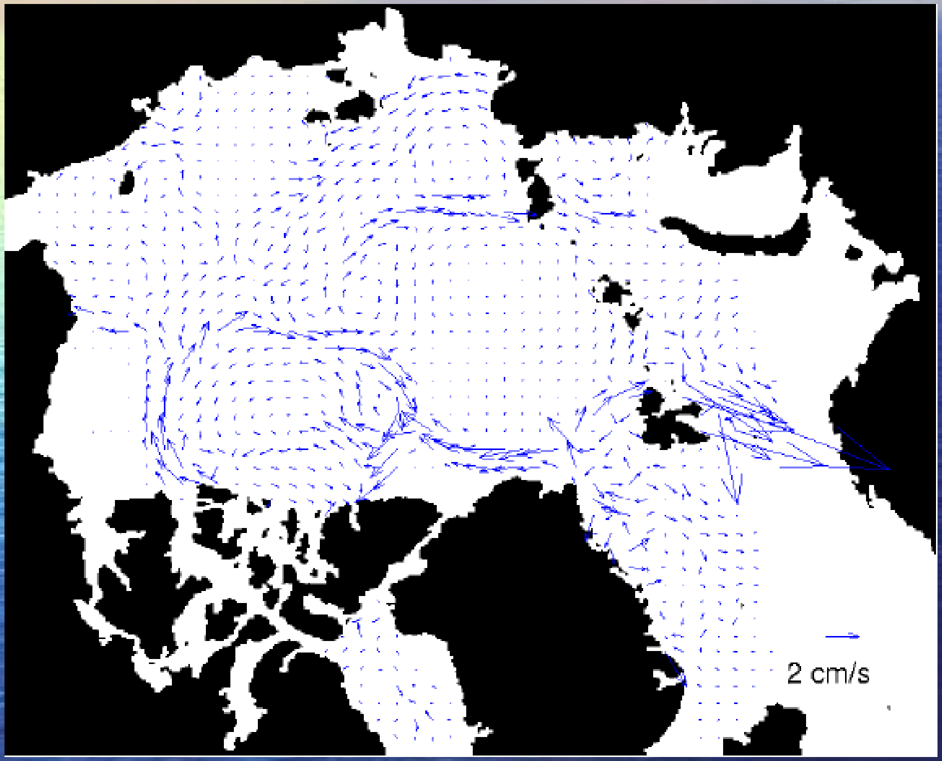


$\Delta t = 6h$

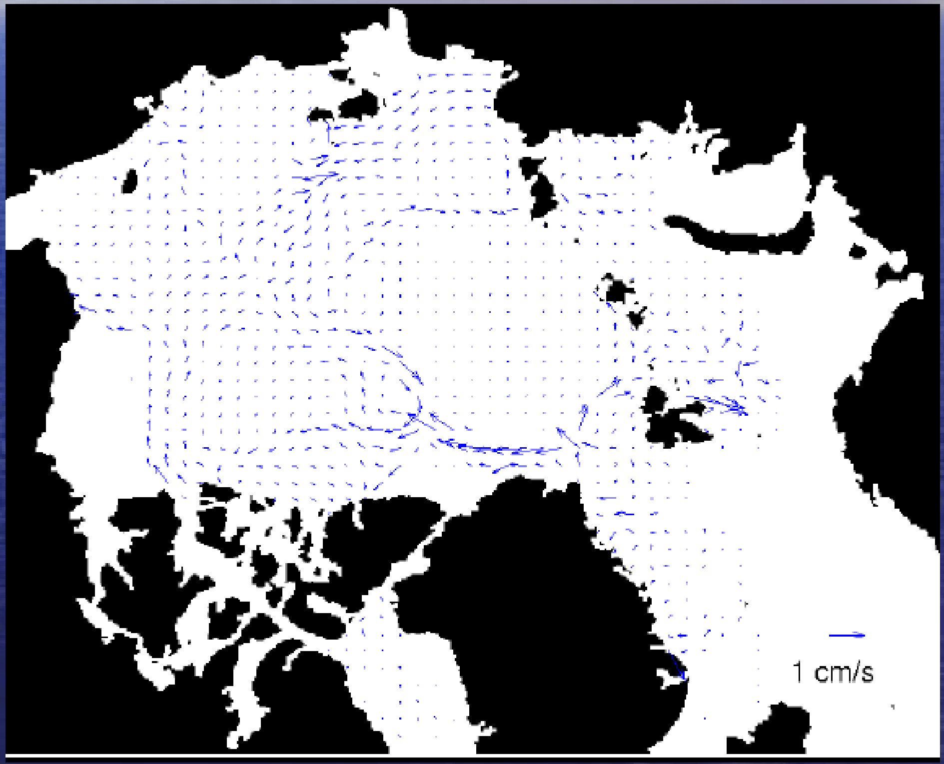
$\Delta x = 10km$

Erreurs après 2 et 10 itérations avec un pas de temps de 30 minutes (6 janvier 1997 00Z)

2 itérations



10 itérations



Δt = 30 min

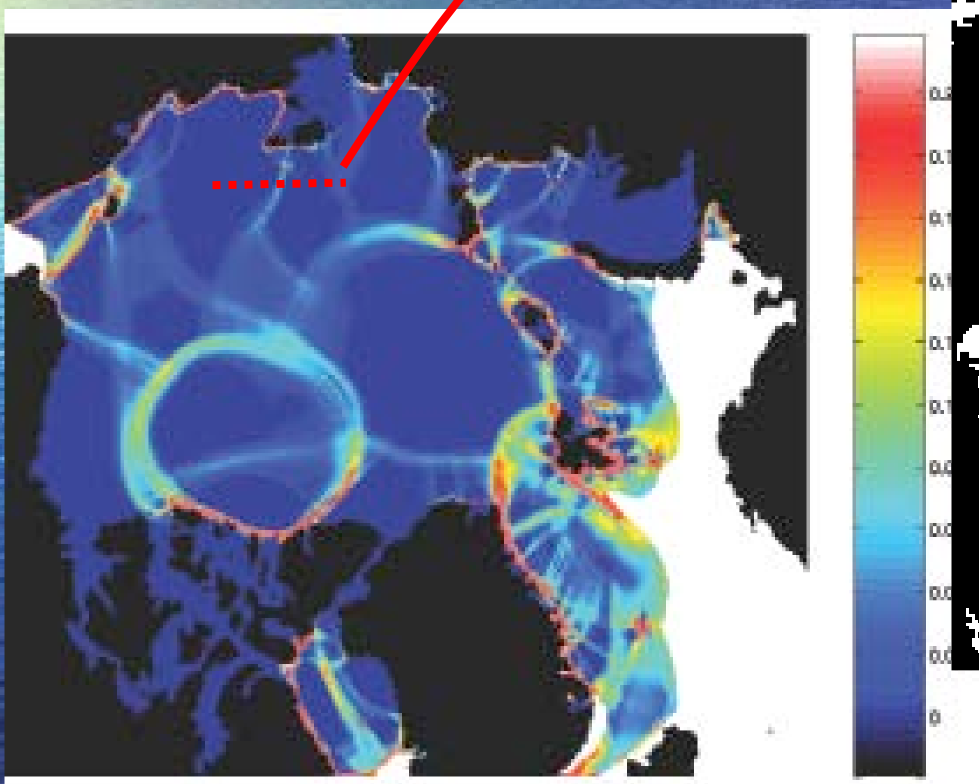
Δx = 10km

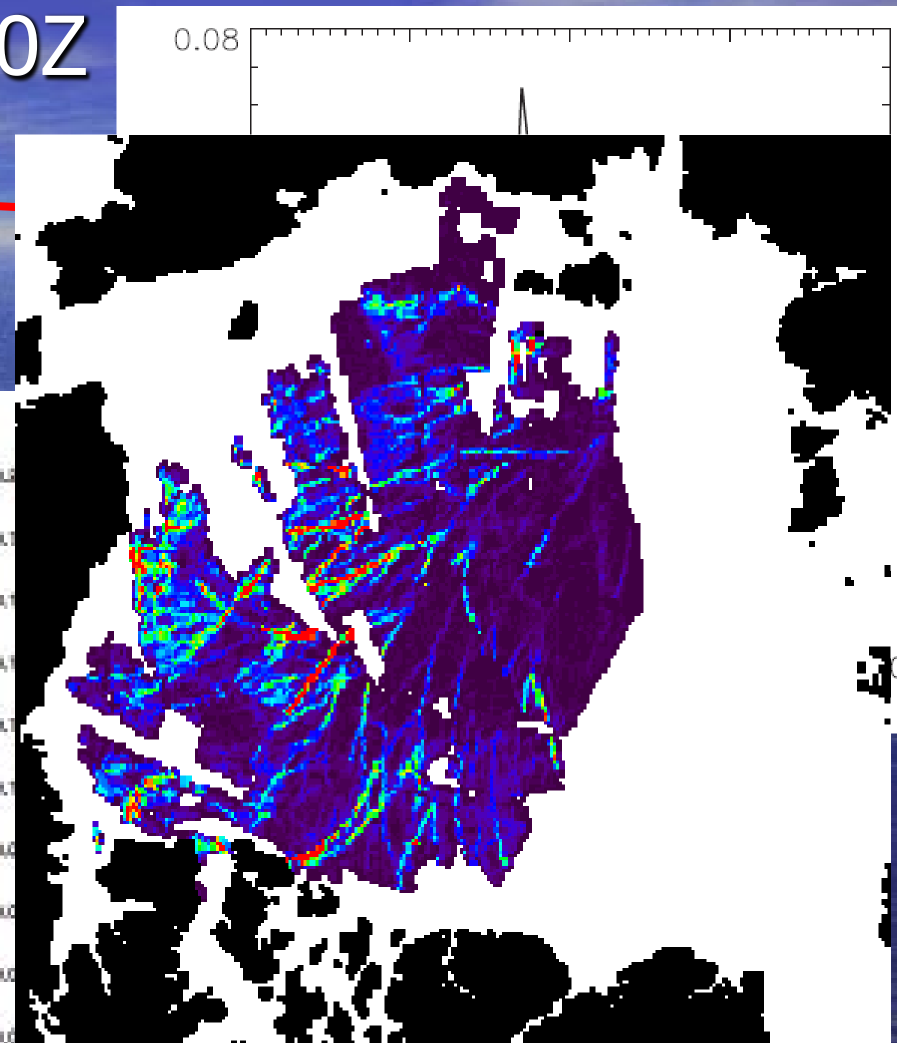
Déformation (cisaillement)

Jan 6 1997 00Z

Δt = 6h

Δx = 10km



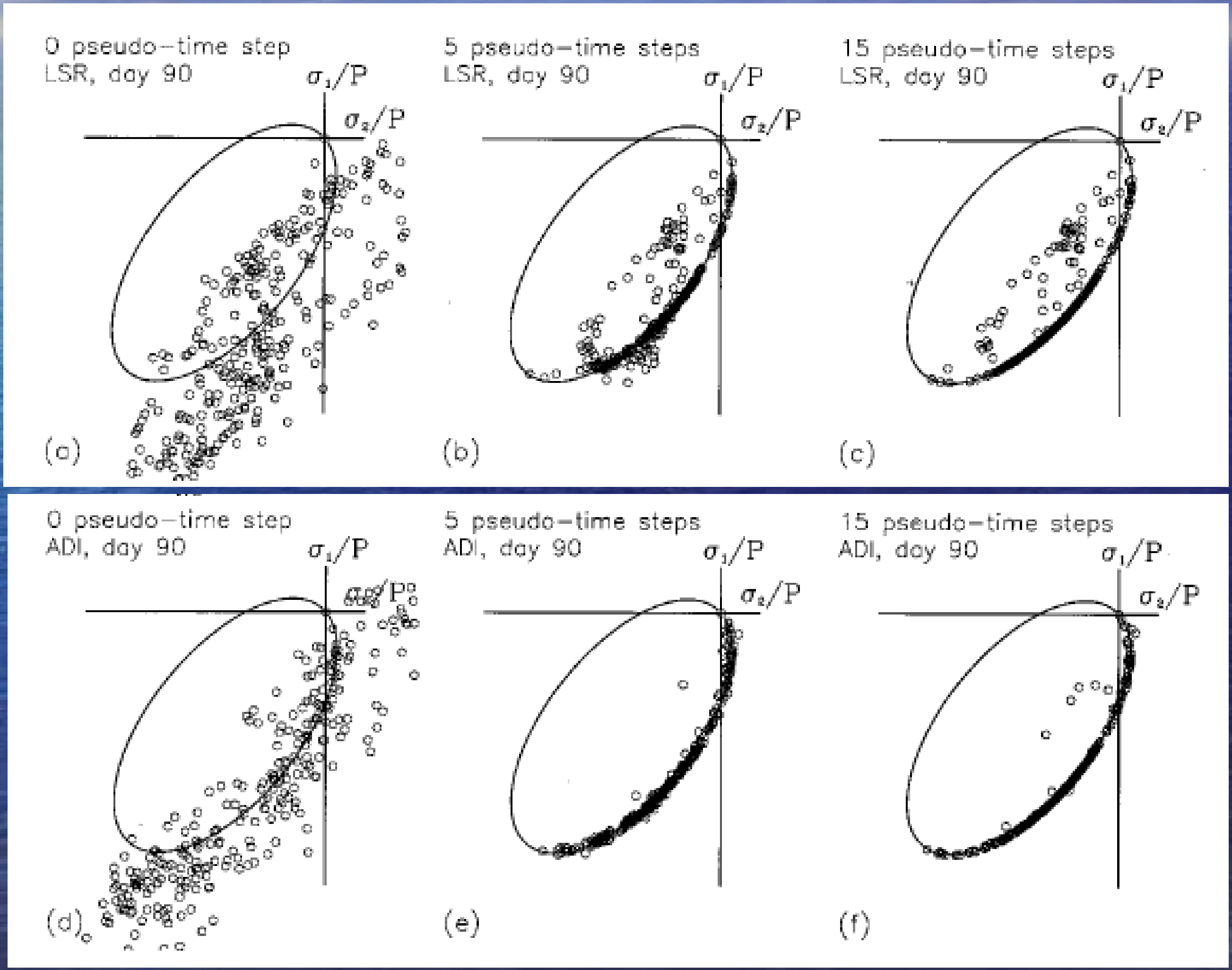


Stess State vs Pseudo Time Step

0 5 15

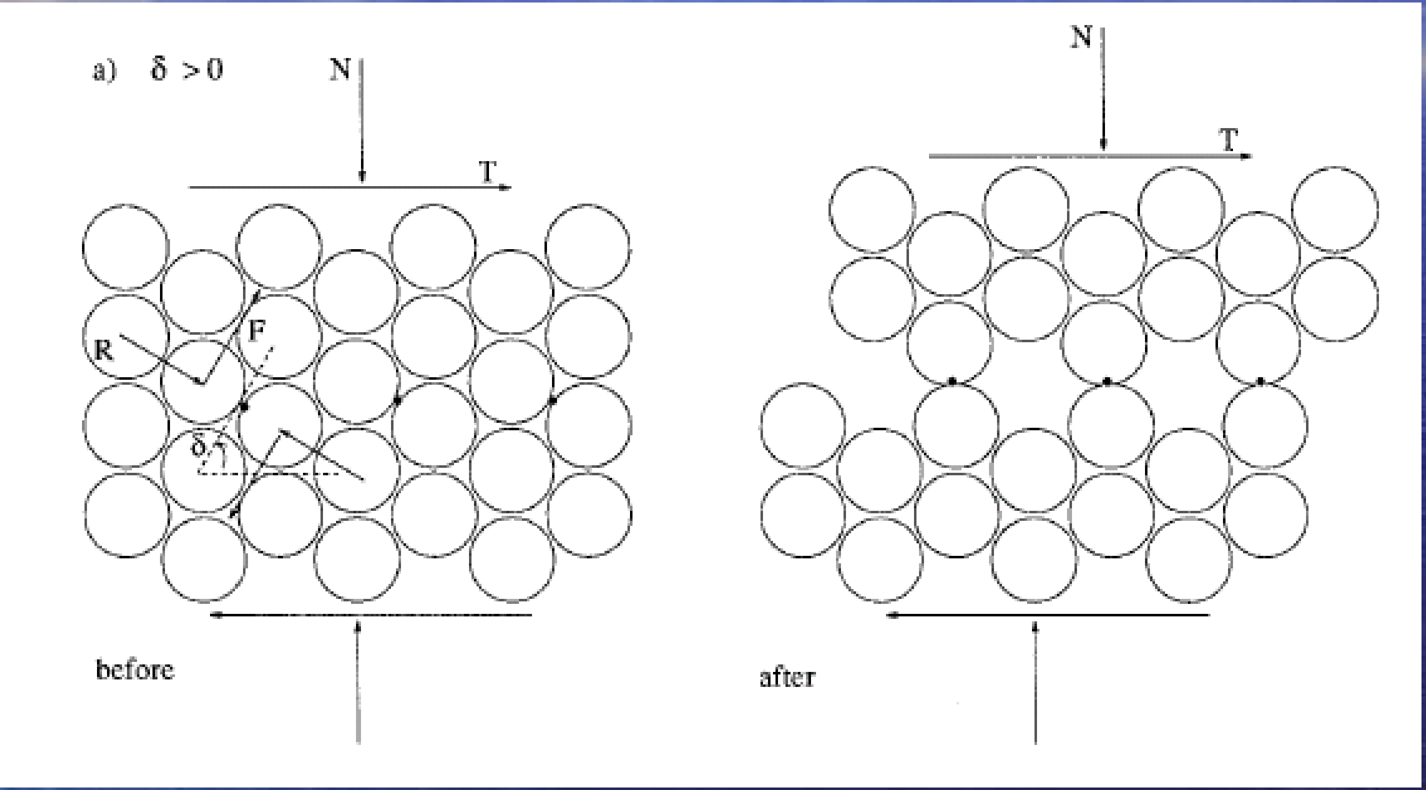
LSR

ADI

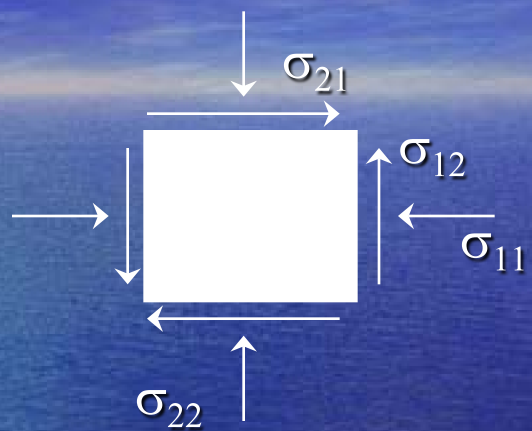


Zhang Hibler, 2000

A very short course on sea ice rheology: the dilatancy effect



Rheology



$$p = -\left(\frac{\sigma_{11} + \sigma_{22}}{2}\right)$$

$$q = \sqrt{\left(\frac{\sigma_{11} - \sigma_{22}}{2}\right)^2 + \sigma_{12}^2}$$

p and q are called the stress invariants

Traitement implicite (iteration k, cut v)

$$-\frac{\rho_i h u^k}{\Delta t} + \rho_i h f v^k_{avg} + \frac{\partial}{\partial x}\left[\left(\eta(u_l)+\zeta(u_l)\right)\frac{\partial u^k}{\partial x}\right] + \frac{\partial}{\partial x}\left[\left(\zeta(u_l)-\eta(u_l)\right)\frac{\partial v^k}{\partial y}\right] +$$

$$\frac{\partial}{\partial y}\left[\eta(u_l)\frac{\partial u^k}{\partial y}\right] + \frac{\partial}{\partial y}\left[\eta(u_l)\frac{\partial v^k}{\partial x}\right] - C_w(u_l)\left(u^k\cos\theta_w - v^k_{avg}\sin\theta_w\right) = b_u$$

$$-\frac{\rho_i h v^k}{\Delta t} - \rho_i h f u^k_{avg} + \frac{\partial}{\partial y}\left[\left(\eta(u_l)+\zeta(u_l)\right)\frac{\partial v^k}{\partial y}\right] + \frac{\partial}{\partial y}\left[\left(\zeta(u_l)-\eta(u_l)\right)\frac{\partial u^k}{\partial x}\right] +$$

$$\frac{\partial}{\partial x}\left[\eta(u_l)\frac{\partial v^k}{\partial x}\right] + \frac{\partial}{\partial x}\left[\eta(u_l)\frac{\partial u^k}{\partial y}\right] - C_w(u_l)\left(v^k\cos\theta_w - u^k_{avg}\sin\theta_w\right) = b_v$$