Modeling the Fracture of Ice Sheets on Parallel Computers

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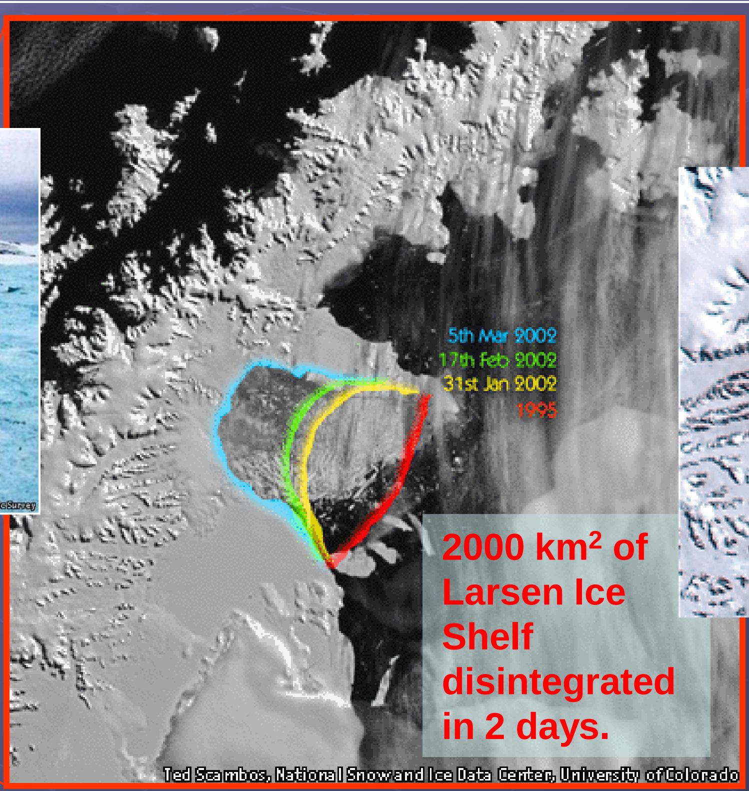
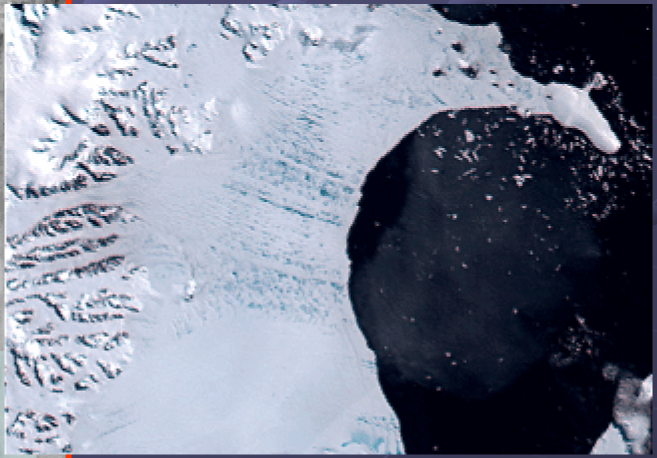
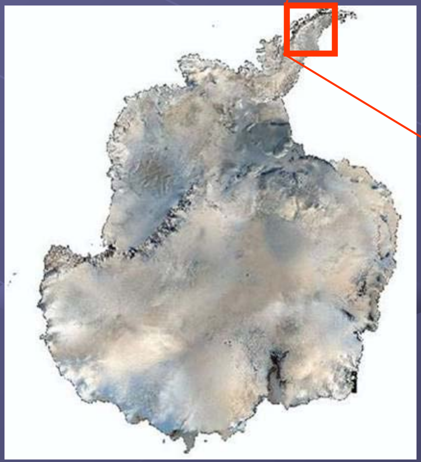
Disintegrating Ice Shelves

Since the mid-1980s





Source: T. Scambos



Require >10,000 years to form

Disintegrate in weeks

Slide from Dr. Bindschadler presentation, DOE workshop Sep. 2009

FEA Model Description

- Geometry
  - Strip model with degrading stiffness towards tip
- Temperature dependent properties
  - Elastic
  - Thermal
- Boundary Conditions
  - Fixed edges
  - Elastic foundation
- Loads
  - Body gravity Load
  - Temperature Increase
- Analysis Steps
  - Static, General with body load
  - Coupled Temp-Disp – steady state

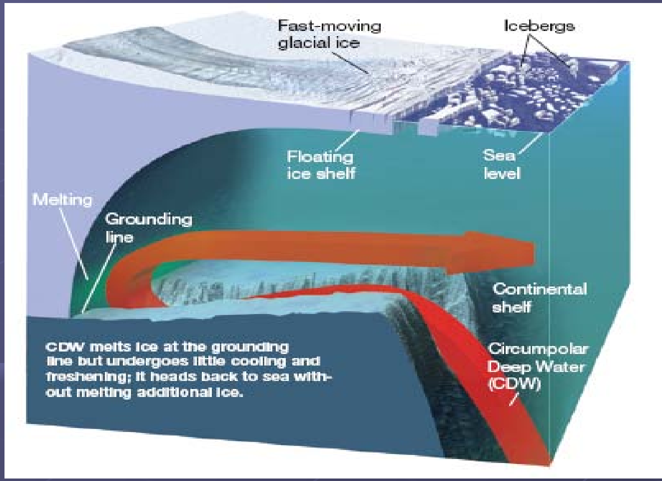
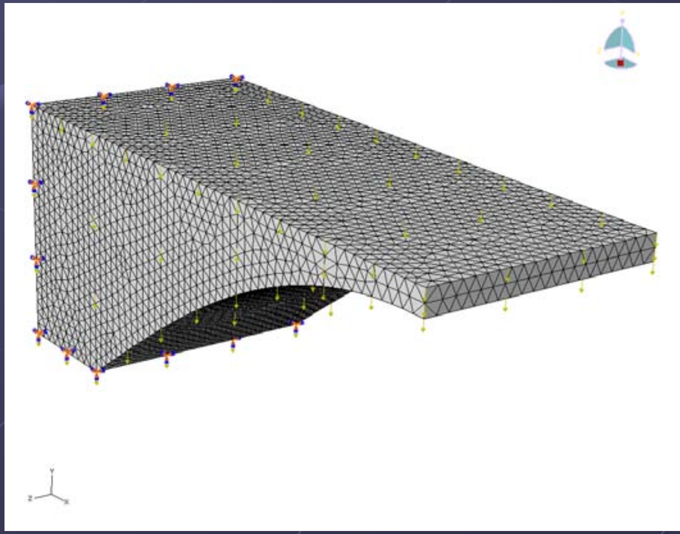


Image from Dr. Bindschadler presentation, DOE workshop Sep. 2009



Basic Equations

Static Force Equilibrium:	**Fourier Law:**
$\sigma_{ij,j} + \mathrm{f}_i = 0$	$\mathrm{q} = -\kappa \nabla \theta$
Constitutive Law:	**Thermal Equilibrium:**
$\sigma_{ij}^{el} = C_{ijkj} \varepsilon_{ij} \equiv \mathrm{D}^{el}(\theta) \cdot \varepsilon$	$-\kappa \nabla^2 \theta + c_p \rho \dot{\theta} = \mathrm{q}$
Deformation:	**Material Properties:**
Elastic: $\varepsilon_{ij}^{el} = \frac{1}{2}(u_{j,i} + u_{i,j})$	$\kappa(\theta)$: Material conductivity
Expansion: $\varepsilon_{ij}^{th} = \alpha \Delta\theta \delta_{ij}$	$c_p(\theta)$: Specific heat capacity
Total-Strain: $\varepsilon_{ij} = \varepsilon_{ij}^{el} + \varepsilon_{ij}^{th}$	$\rho(\theta)$: density

Note: Since the time scales for (a) temperature change (which affects both the elastic and thermal properties) and (b) thermal equilibrium are vastly different, these two sets of conditions are only weakly coupled and can be solved independently

Ice – Physical, Thermal and Mechanical Properties

Temperature	Density	Thermal conductivity	Specific Heat
T (c)	rho (kg/m^3)	K (W/m.K)	Cp (J/kg.K)
0	916.2	2.22	2.05E-03
-5	917.5	2.25	2.03E-03
-10	918.9	2.30	2.00E-03
-15	919.4	2.34	1.97E-03
-20	919.4	2.39	1.94E-03
-25	919.6	2.45	1.91E-03
-30	920.0	2.50	1.88E-03
-35	920.4	2.57	1.85E-03
-40	920.8	2.63	1.82E-03
-50	921.6	2.76	1.75E-03
-60	922.4	2.90	1.68E-03
-70	923.3	3.05	1.61E-03
-80	924.1	3.19	1.54E-03
-90	924.9	3.34	1.46E-03
-100	925.7	3.48	1.39E-03

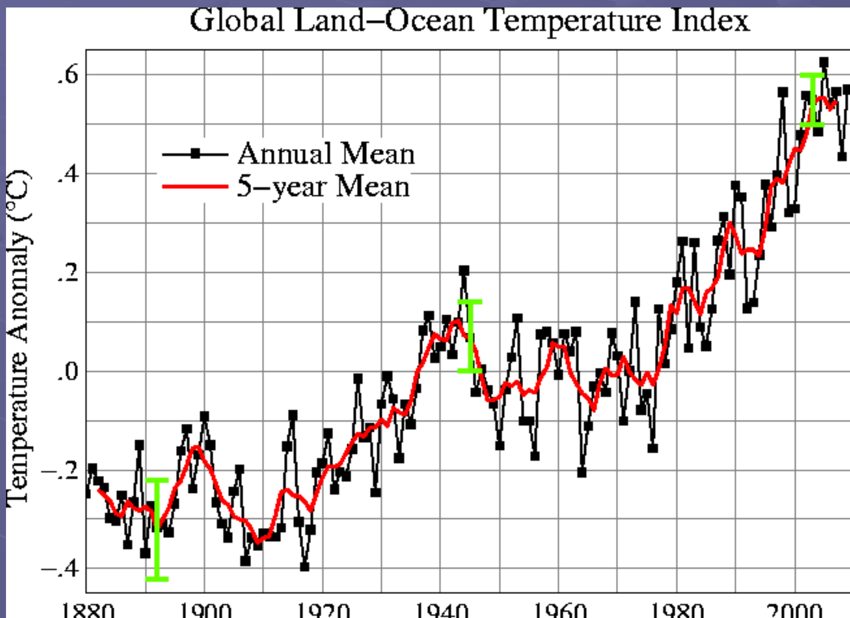
Elastic Modulus	Temperature
E (Gpa)	Temp (C)
6.0	0
7.0	-10
7.5	-20
8.0	-30
8.5	-40
9.0	-50
10.0	-60
11.0	-80
12.0	-100

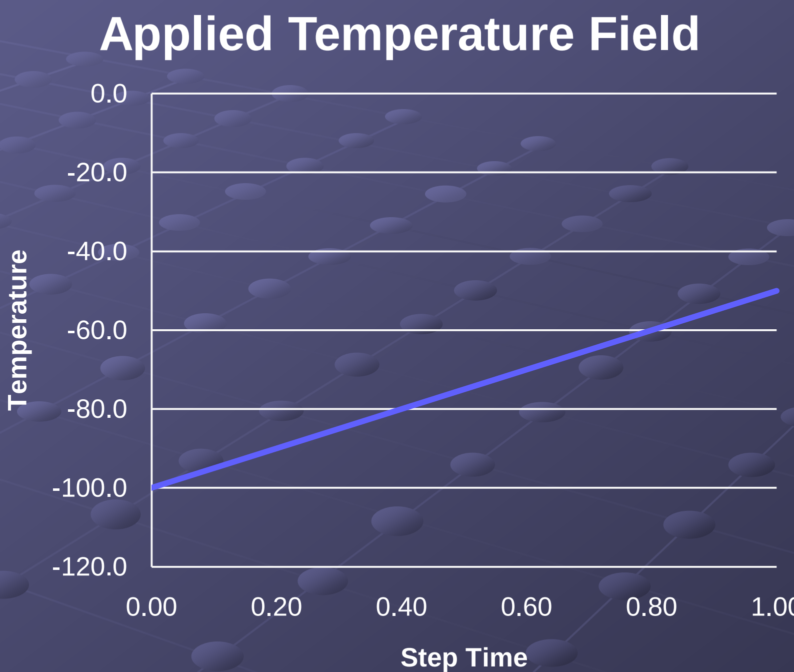
The elastic modulus properties have a wide range of variability and is generally strongly orthotropic. In the preliminary analysis, it was assumed to be linearly isotropic with softening at higher temperature

Thermal Expansion
α = 5.1E-05 /deg. C

References:
http://www.engineeringtoolbox.com/ice-thermal-properties-d_576.html
http://www.its.caltech.edu/~atomic/snowcrystals/ice/ice.htm

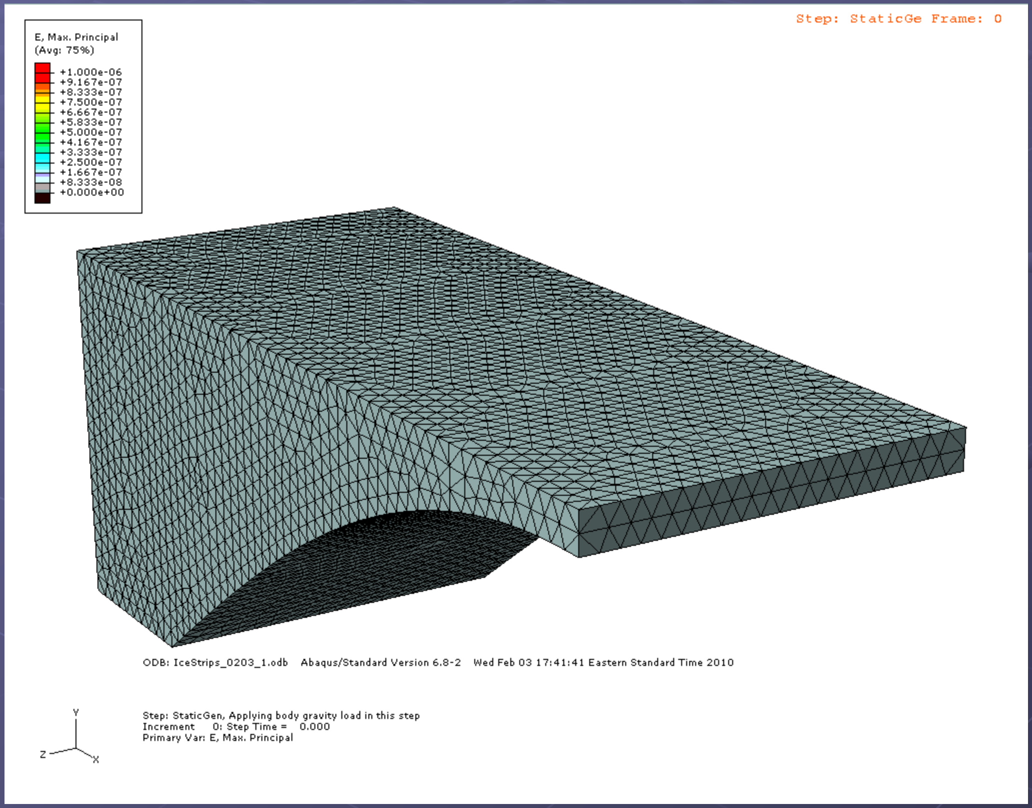
Temperature Rise



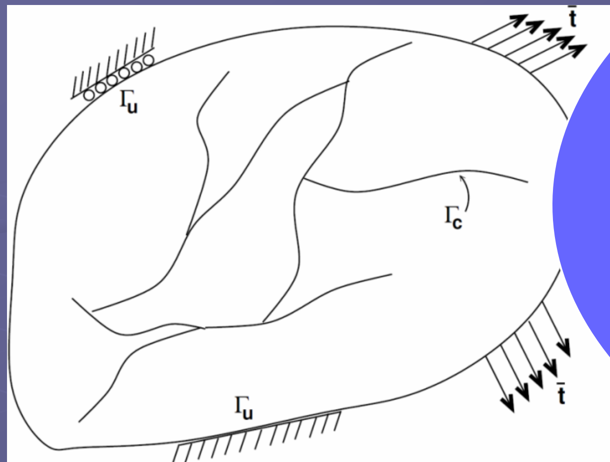


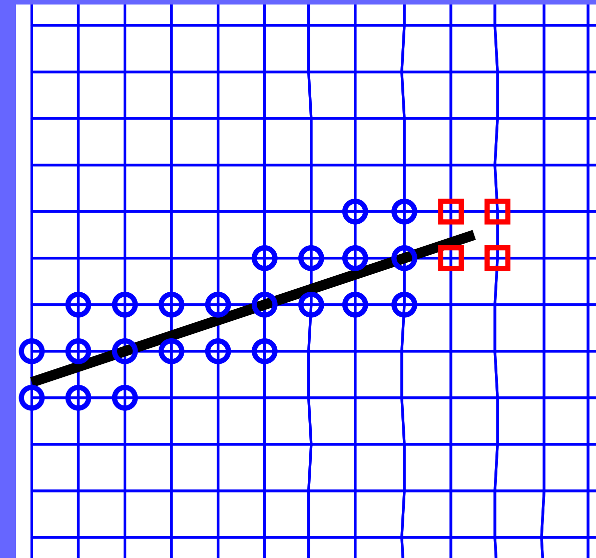
Reference:
Hansen, J., Mki. Sato, R. Ruedy, K. Lo, D.W. Lea, and M. Medina-Elizade, 2006: Global temperature change. *Proc. Natl. Acad. Sci.*, 103, 14288-14293, **http://data.giss.nasa.gov/gistemp/graphs/**

Preliminary Simulation



How XFEM works





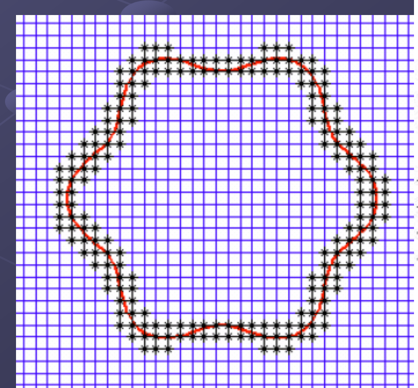
Displacements written as

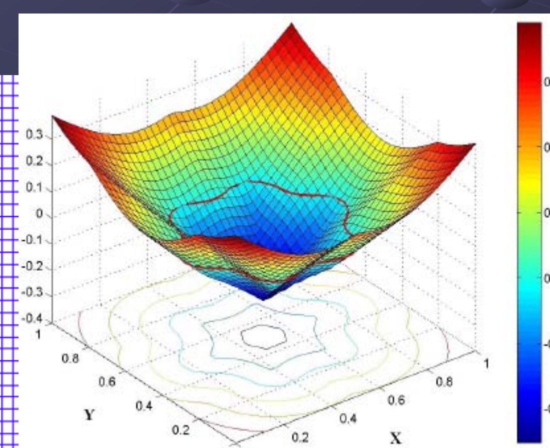
$$u^h(\boldsymbol{x}) = \sum_{I=1}^{n} N_I(\boldsymbol{x}) u_I + \sum_{I=1}^{n_J} N_I(\boldsymbol{x}) H(\boldsymbol{x}) a_I + \sum_{I=1}^{n_T} \left[N_I(\boldsymbol{x}) \sum_{j=1}^{4} F_j(\boldsymbol{x}) b_{jI} \right]$$

Heaviside jump-enrichment

$$H(\mathbf{X}) = \begin{cases} 1 & \text{above } \Gamma_c^+ \\ 0 & \text{bellow } \Gamma_c^- \end{cases}$$

Levelset Method

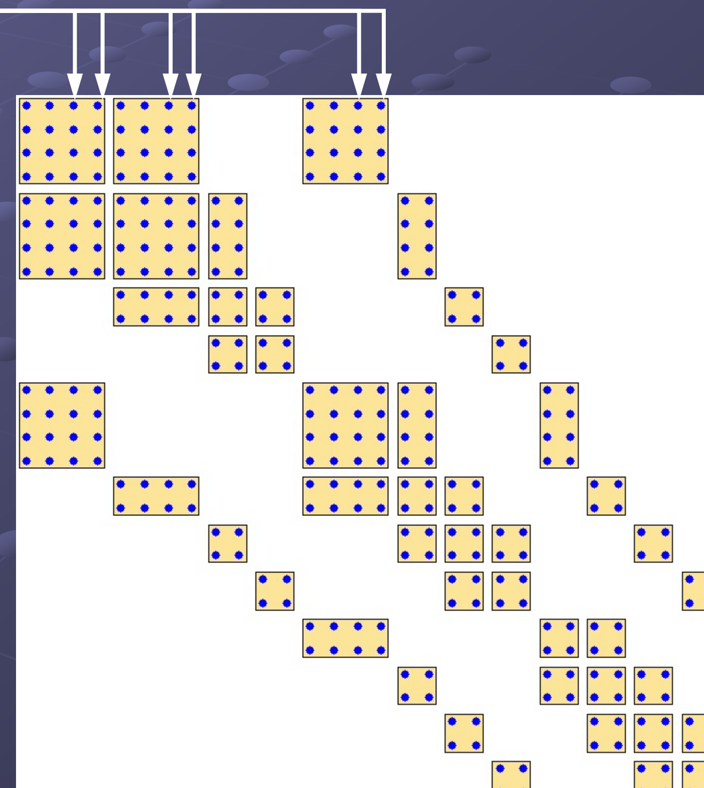




Belytschko et. al., IJNME [1999]

XFEM Linear System

$$u(x) = \sum_i \phi_i(x) u_i + \sum_i \phi_i(x) H(x) z_i + \ldots \quad , \quad H(x) = \begin{cases} 1 & \varphi \geq 0 \\ 0 & \varphi < 0 \end{cases}, \quad \Gamma = \{(x,y) \mid \varphi(x,y) = 0\}$$



- **Very large linear system**
- **Variable block size matrix with peculiar basis functions**

Alternative view of matrix:

$$A = \begin{pmatrix} A_{ss} & A_{sw} \\ A_{ws} & A_{ww} \end{pmatrix}$$

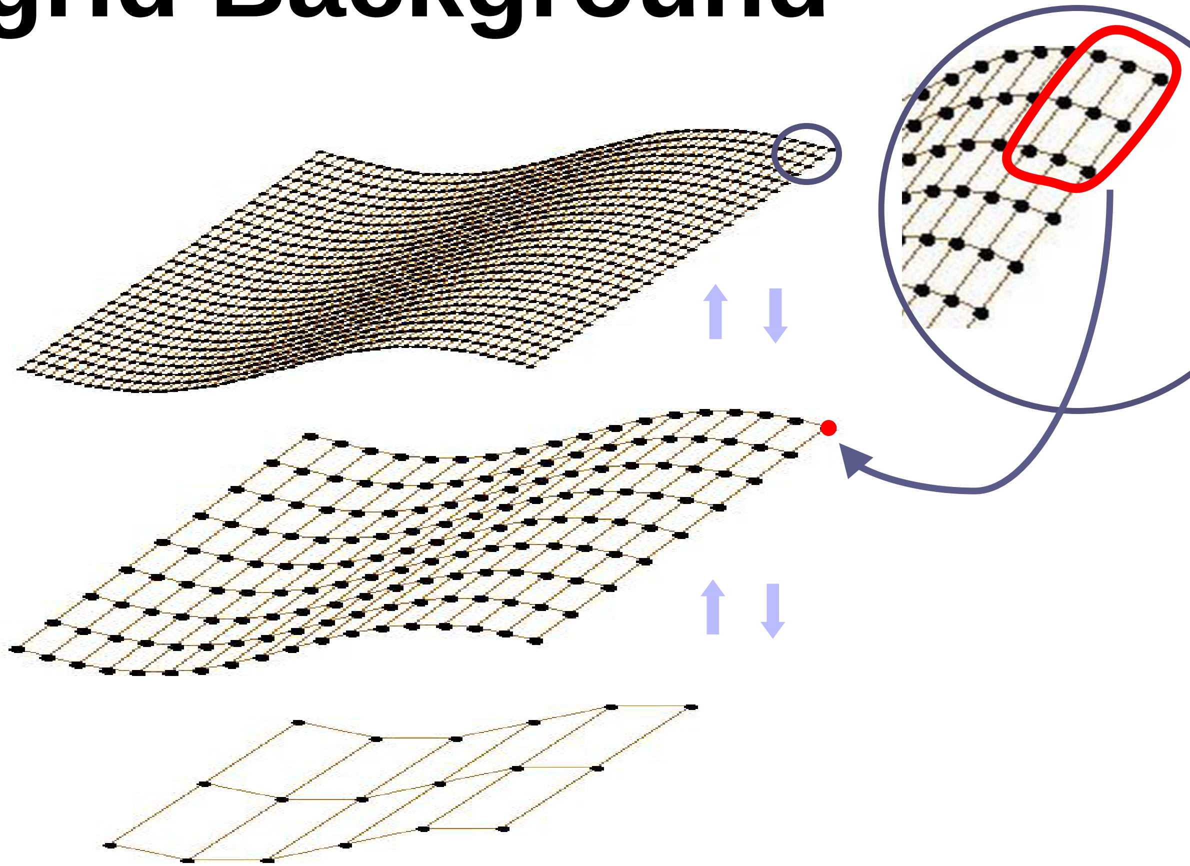
standard → A_{ss}

weird → A_{ww}

$\Rightarrow$ **Need fast highly parallel solver !**

Algebraic Multigrid Background

Solve $A_3 u_3 = f_3$

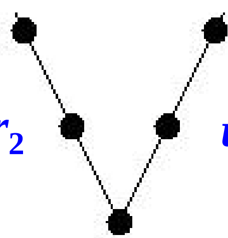


- **Construct graph & coarsen**
- **Determine P_i sparsity pattern**
- **Determine P_i 's coefs**
- **Project: $A_i = (P_i)^T A_{i+1} P_i$**

Smooth $A_3 u_3 = f_3$, $f_2 \leftarrow (P_2)^T r_3$

Smooth $A_2 u_2 = f_2$, $f_1 \leftarrow (P_1)^T r_2$

Solve $A_1 u_1 = f_1$ directly



$u_3 \leftarrow u_3 + P_2 u_2$

$u_2 \leftarrow u_2 + P_1 u_1$

Brandt, McCormick, Ruge, Stubën

AMG & Energy Minimization

Basic Idea

A_k
P_k sparsity pattern
mode constraints

→

minimize energy
while
satisfying constraints

→

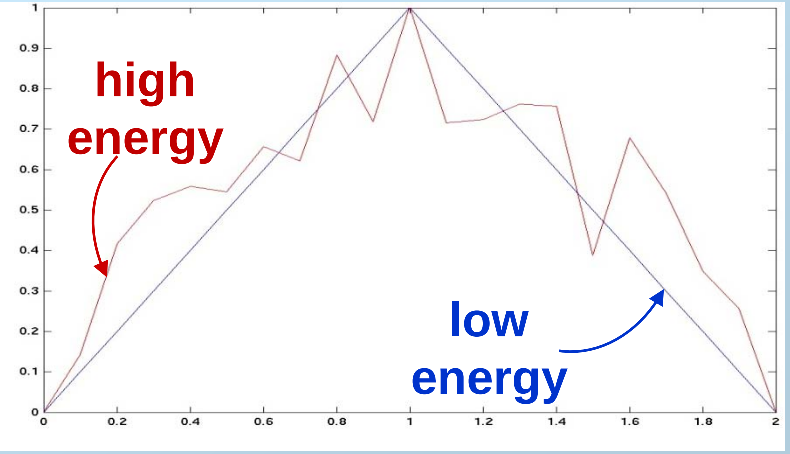
P_k's
coefficients

Tradeoffs:

+ **flexibility**
  - **any coarsening**
  - **any sparsity pattern**
  - **constraints**
    - **<u>important modes</u>** **requiring accurate interpolation**

+ **robustness**

Constraints : accurate interpolation of certain modes, e.g. constant



XFEM philos

Standard FEM requires special meshes to model flaws

standard grid

standard basis functions

\+ weir

Our AMG philosophy

standard coarsening

standard prolongator columns

\+ weird prolongator columns

$$A = \begin{pmatrix} A_{ss} & A_{sw} \\ A_{ws} & A_{ww} \end{pmatrix}$$

$$P = \begin{pmatrix} P_{ss} & P_{sw} \\ 0 & P_{ww} \end{pmatrix}$$

P_{ss} ← StandardAMG(A_{ss})

P_{ww} ← I (currently not coarsening special functions)

P_{sw} use standard energy minimization

- determine sparsity pattern
- special interpolation constraints

Details

Energy Minimization

- **Boils down to repeated iteration on**

$$A_{ss} P_{sw} = -A_{sw}$$

followed by sparsity restriction & enforcing mode constraints

- **Without sparsity limits**

$$\Rightarrow P_{sw} = -(A_{ss})^{-1} A_{sw} \quad \textbf{(Schur complements)}$$

P_{sw} Sparsity Pattern

$$\mathcal{N} = p(A_{ss}) A_{sw}$$

for experiments $p(A_{ss}) = A_{ss}$

Special constraints

- **3 zero energy modes corresponding to discontinuity captured by XFEM**

Special Constraints

Standard AMG

Special Constraints

Standard AMG & XFEM

Only need to add special constraints near crack

Results for 1 crack

$$P = \begin{pmatrix} P_{ss} & Z \\ 0 & I \end{pmatrix}$$

n x n grid	AMG on A_{ss}	*Z=0* AMG	*Z ≠ 0* AMG	*Z ≠ 0* AMG + constraints
6	9	9	8	8
18	9	13	10	9
54	9	19	14	9
162	10	29	21	9

crack crosses domain, aligned coarsening

n x n grid	AMG on A_{ss}	*Z = 0* AMG	*Z ≠ 0* AMG	*Z ≠ 0* AMG + constraints
6	9	11	11	10
18	9	14	12	10
54	9	21	15	11
162	10	32	23	11

partial domain crossing, coarsening aligned

n x n grid	AMG on A_{ss}	*Z=0* AMG	*Z ≠ 0* AMG	*Z ≠ 0* AMG + constraints
6	9	10	9	8
18	9	16	11	9
54	9	25	16	10
162	10	43	24	12

crack crosses domain, nonaligned coarsening

n x n grid	AMG on A_{ss}	*Z = 0* AMG	*Z ≠ 0* AMG	*Z ≠ 0* AMG + constraints
6	9	13	11	11
18	9	17	13	11
54	9	29	18	13
162	10	50	27	15

partial crossing, nonaligned coarsening

Conclusions

- Initial studies to explain the collapse (disintegration) of an ice shelf with simple models
- Ice is a complicated material
  - Further research is needed to characterize the material behavior
- The model may provide the crack initiation for the fracture modeling
- Algebraic multigrid can mirror XFEM idea !
  - Special prolongator columns to capture discontinuities
- Promising results, but …
  - Still need to resolve special constraints near crack tip & determine special constraints for crack network