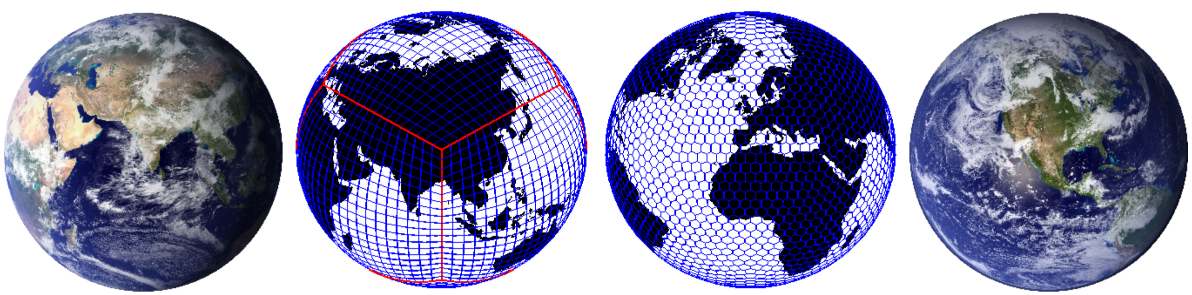
The CSLaM transport scheme
&
new numerical mixing diagnostics

CESM Ocean Model Working Group Meeting
14 – 15 December, 2011

Peter Hjort Lauritzen

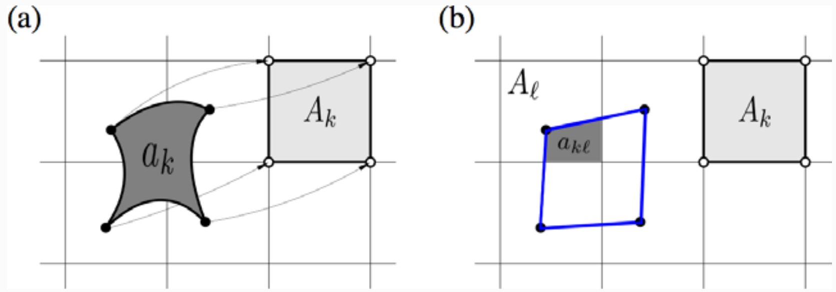


Climate & Global Dynamics

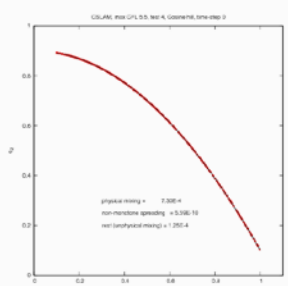
NCAR Earth System Laboratory

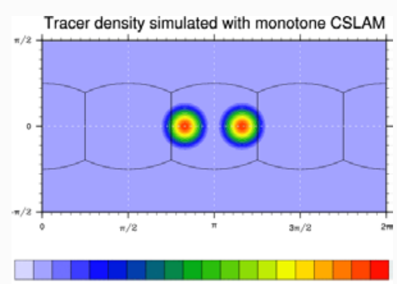
Outline: Two parts

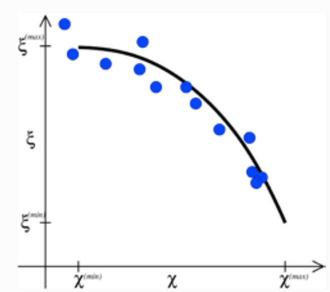
Briefly introduce the CSLaM scheme
(Conservative Semi-Lagrangian Multi-tracer)



Evaluating advection/transport schemes using interrelated tracers, scatter plots and numerical mixing diagnostics





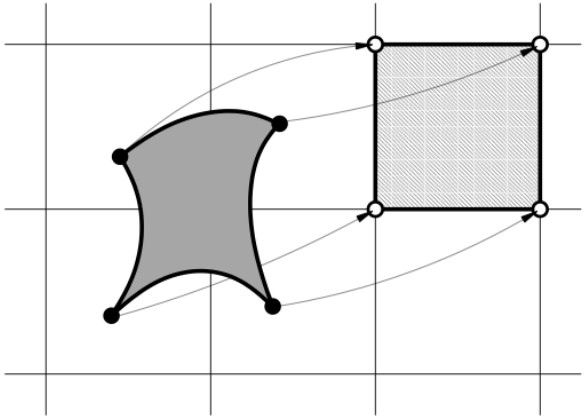


A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLaM)

Consider two-dimensional transport equation for an inert and passive tracer:

$$\frac{d}{dt}\int_{A(t)} \psi \, dA = 0$$

where ψ density and $A(t)$ arbitrary Lagrangian area (e.g., Machenhauer et al, 2009).



Lauritzen et al. (2010)

A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLaM)

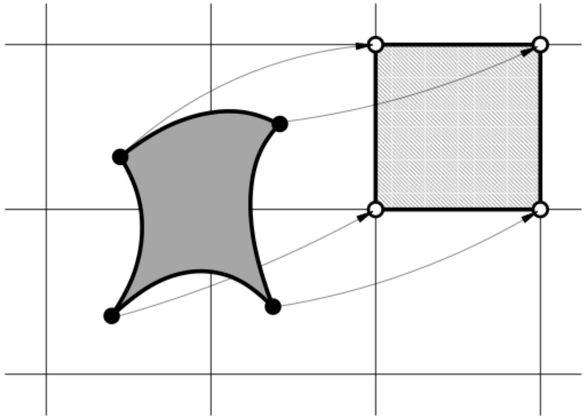
Consider tw   rt and passive tracer:

Stability criterion:

upstream cell must be simply-connected

i.e. CSLaM allows for large time-steps!

where ψ density and $A(t)$ arbitrary Lagrangian area (e.g., Machenhauer et al, 2008).



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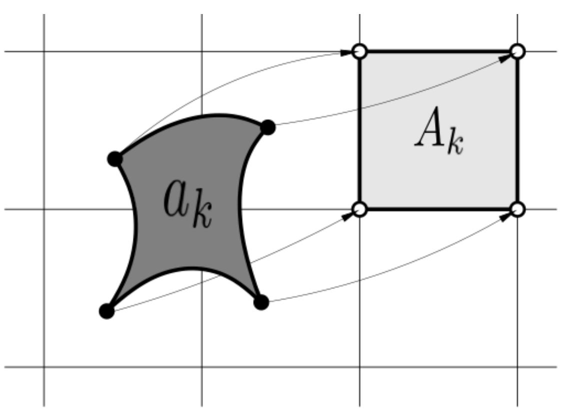
A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLaM)

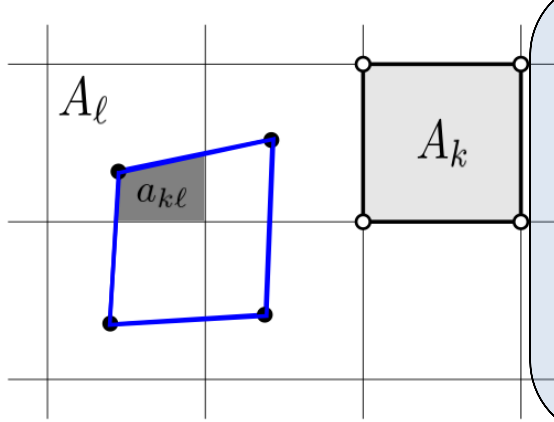
Consider two-dimensional transport equation for an inert and passive tracer:

$$\int_{A(t+\Delta t)} \psi \, dA = \int_{A(t)} \psi \, dA = \sum_{\ell=1}^{L_k} \iint_{a_{k\ell}} f_\ell(x,y)\, dx\, dy$$

where overlap areas are

$$a_{k\ell} = a_k \cap A_\ell, \quad a_{k\ell} \neq \emptyset; \quad \ell = 1, \ldots, L_k$$





Cell sides = straight lines (great-circle arcs) *Lauritzen et al. (2010)*

Curves (parabolic) cell sides *(Ullrich et al., 2011, JCP, submitted)*

A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLaM)

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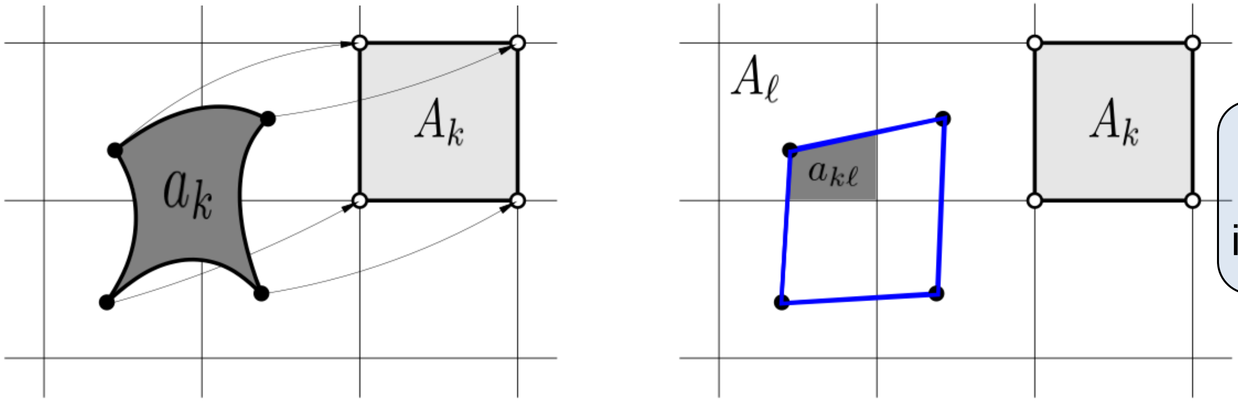

Use **Gauss-Green's** theorem to convert area integrals into line-integrals.

Lauritzen et al. (2010)

A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLaM)

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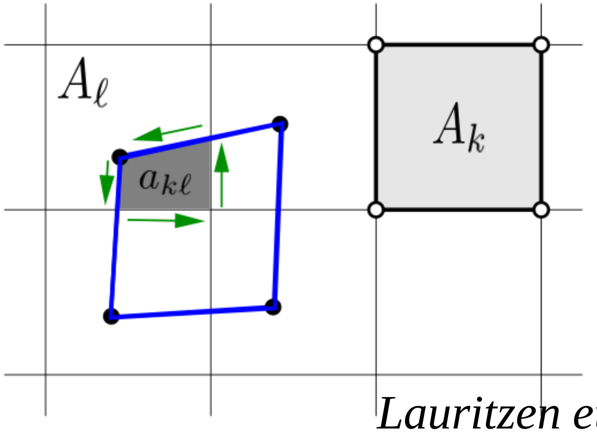
$$\int_{A(t+\Delta t)} \psi \, dA = \int_{A(t)} \psi \, dA = \sum_{\ell=1}^{L_k} \oint_{\partial a_{k\ell}} \left[P\, dx + Q\, dy \right],$$

where P, Q are potentials so that

$$-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} = f_\ell(x, y)$$



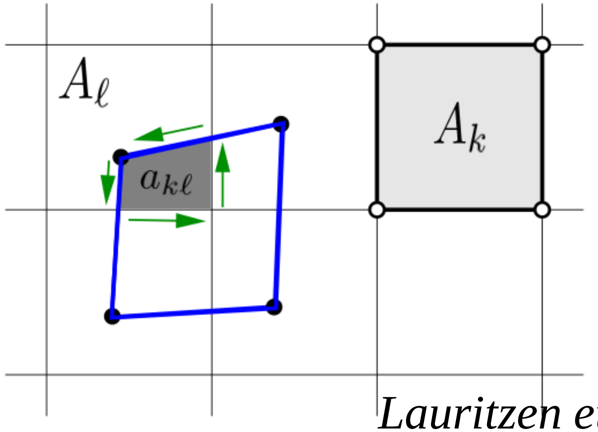
Lauritzen et al. (2010)

A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLaM)

Final **CSLaM** forecast equation becomes:

$$\int_{A(t+\Delta t)} \psi \, dA = \int_{A(t)} \psi \, dA = \sum_{\ell=1}^{L_k} \left[\sum_{i+j \leq 2} c_\ell^{(i,j)} w_{k\ell}^{(i,j)} \right],$$

where weights $w_{k\ell}^{(i,j)}$ are functions of the coordinates of the vertices of the overlap areas $a_{k\ell}$



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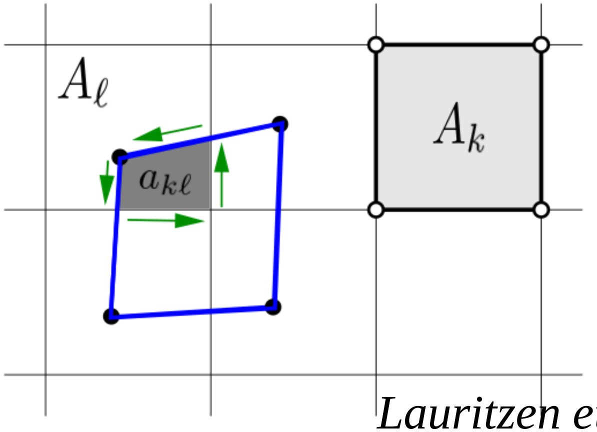
where weights $w_{k\ell}^{(i,j)}$ are functions of the coordinates of the vertices of the overlap areas $a_{k\ell}$

$w_{k\ell}^{(i,j)}$ can be reused for each additional tracer
(Dukowicz and Baumgardner, 2000)



Lauritzen et al. (2010)

A Conservative Semi-Lagrangian Multi-Tracer Transport Scheme (CSLaM)

- in theory CSLaM can accommodate any grid constructed for great-circle arcs
- Gauss-Green's theorem must be extended to spherical projection
- for gnomonic projection (cubed-sphere) the math is "beautiful" (line-integrals along lines parallel to grid-lines are analytic!)
- with fully 2D bi-parabolic reconstructions we get third-order convergence on equi-angular cubed-sphere (slightly less for shape-preserving transport)

CSLaM is being implemented into CAM-SE (2D implementation almost complete)

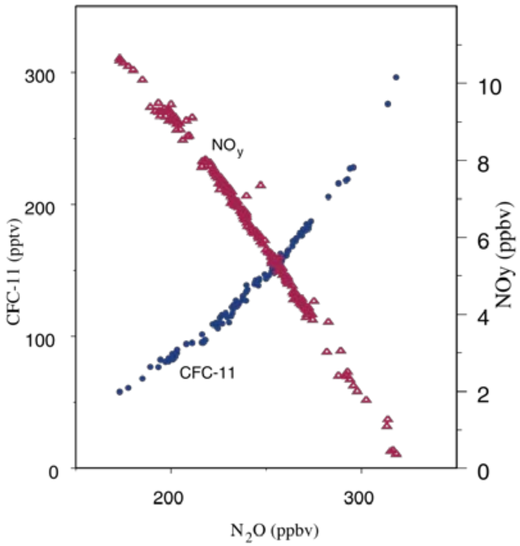
CSLAM can also be cast in flux-form (Harris et al., 2011).
- permits use of flux-limiters such as FCT
- mass-conservation is trivial (more complicated for Lagrangian version)
- minimum 40% more expensive than Lagrangian formulation (for one tracer)
- rigorous search for overlap areas can be eliminated for CFL<1 (only use on reconstruction per face; Lauritzen et al. 2011)
- flux-form version is based on incremental remapping scheme (Dukowiczand and Baumgardner, 2000) but extended for higher-order, CFL>1, …

New physically motivated methodology for testing transport schemes

Atmospheric tracers are often observed to be functionally related, and these relations can be physically or chemically significant.

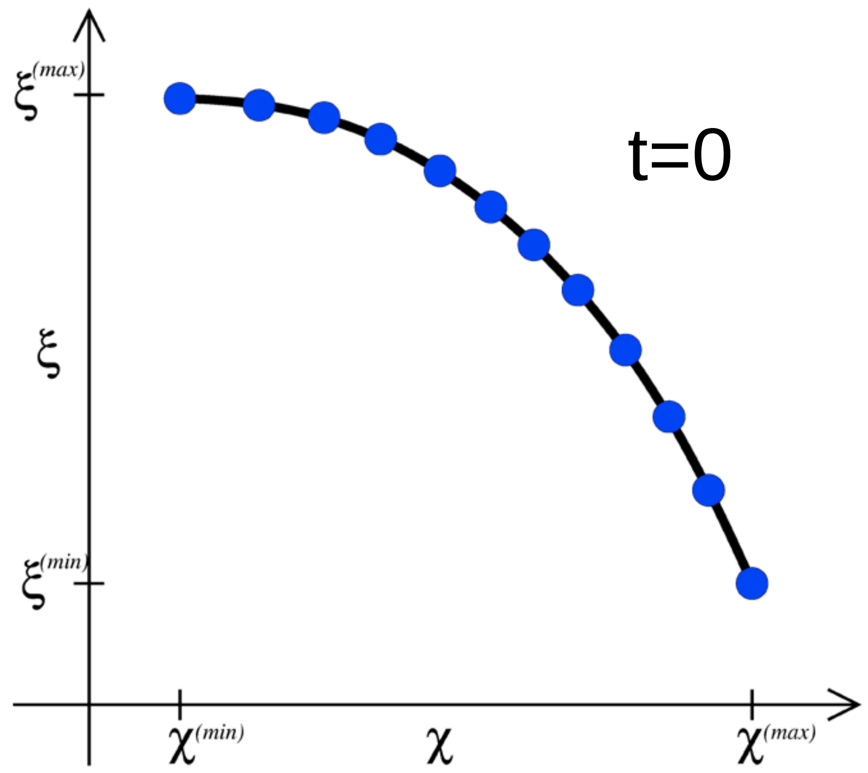
Transport schemes used in chemistry and chemistry-climate models should not disrupt such functional relations in unphysical ways through numerical mixing or, indeed, unmixing.

Might also be relevant for ocean modeling?

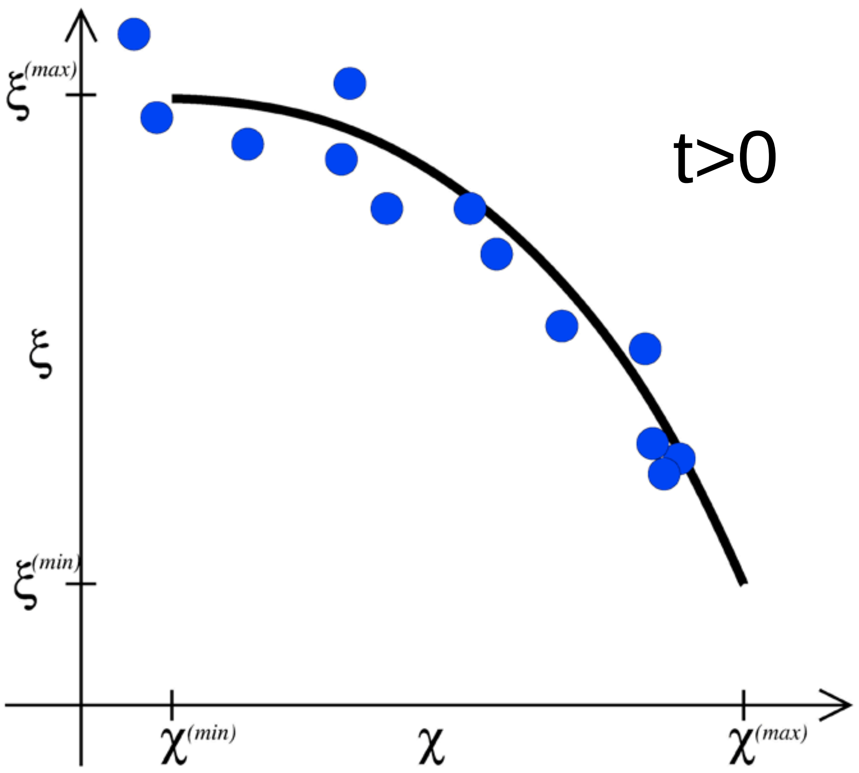


Stratospheric observations (Plumb, 2007)

Consider two inert and passive tracers that are non-linearly related and transported by some flow field

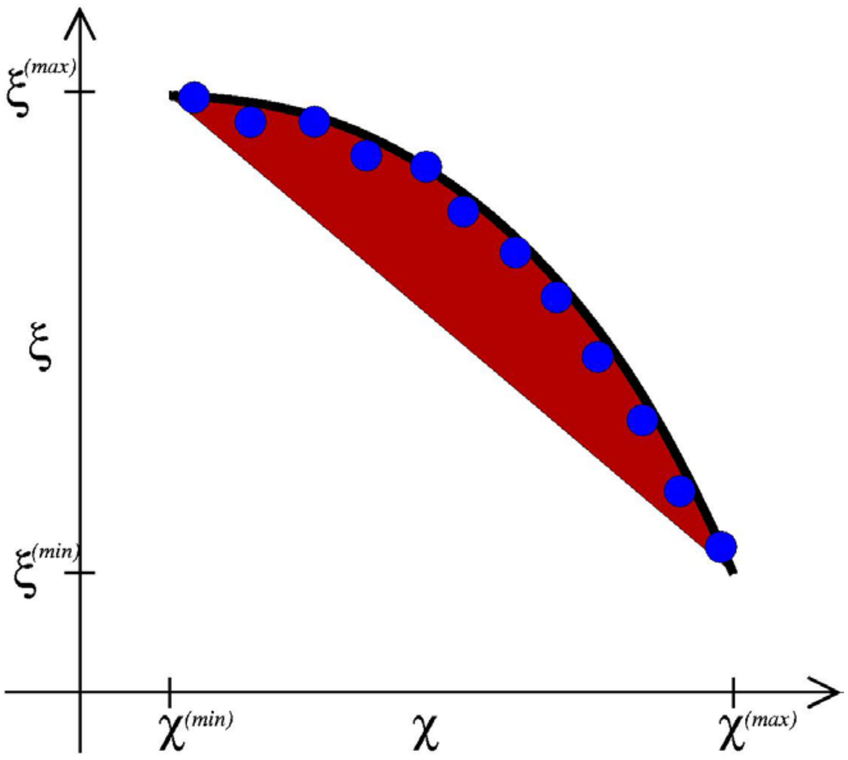


If scheme does not introduce any mixing the scatter plot should not change in time (e.g., fully Lagrangian schemes without explicit diffusion operators).



Any Eulerian/semi-Lagrangian scheme will produce mixing between the tracers!

How physically reasonable is the numerical mixing between the tracers?

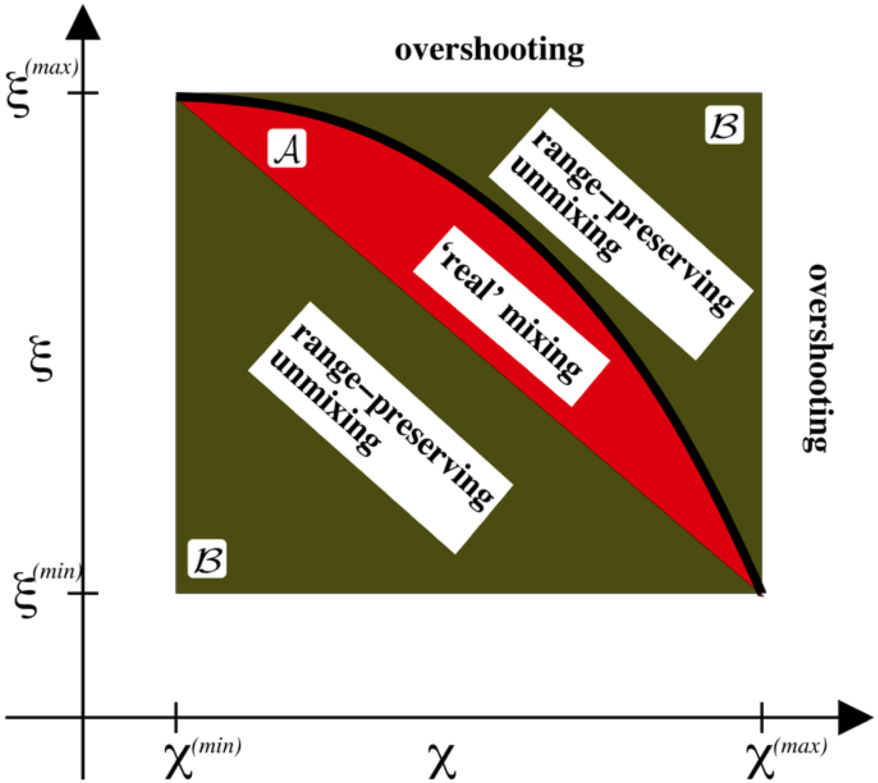


‘Real mixing’ = scatter points shifted into convex hull

Resembles mixing as observed in nature

Only first-order schemes can guarantee that all mixing is ‘real mixing’

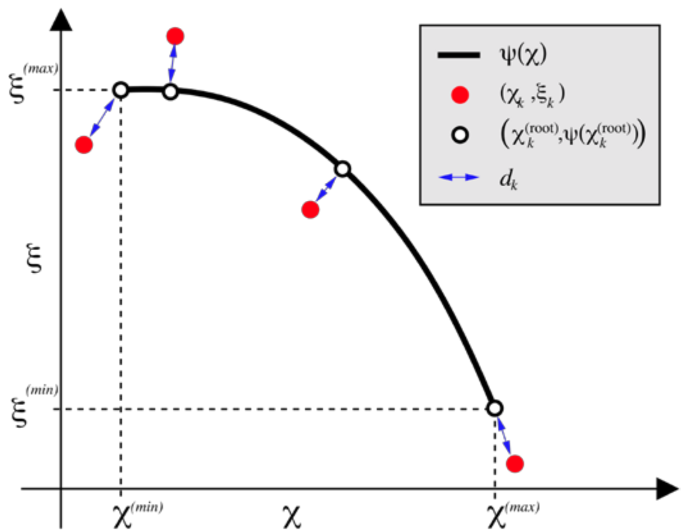
Classification of numerical mixing



'Range-preserving unmixing' = this diagnostic accounts for numerical unmixing within the range of the initial data (spurious)

'Overshooting' = range-expanding unmixing ("very" spurious)

Quantification of mixing



Mixing is quantified in terms of normalized distance function d_k , e.g., for scatters point in the convex hull:

$$\ell_r = \frac{1}{A} \sum_{k=1}^{K} \begin{cases} d_k \, \Delta A_k, & \text{if } (\chi_k, \xi_k) \in \mathcal{A}, \\ 0, & \text{else,} \end{cases}$$

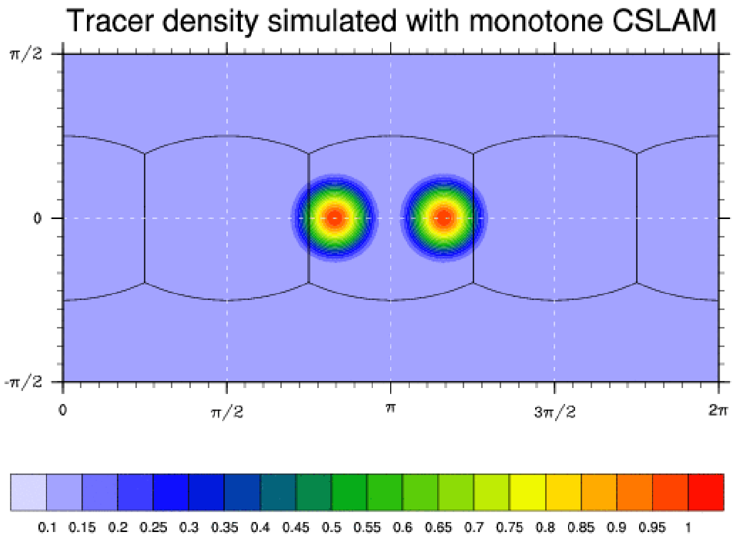
Note: exact/reference solution is not needed for mixing diagnostics!

For simplicity: Use analytical flow field

(Nair and Lauritzen, 2010)

It is key that tracer features collapse to smaller scales (as in nature)

$$u(\lambda, \theta, t) = \kappa \sin(2\lambda') \sin(2\theta) \cos(\pi t/T) + 2\pi \cos(\theta)/T$$
$$v(\lambda, \theta, t) = \kappa \sin(2\lambda') \cos(\theta) \cos(\pi t/T),$$
$$\lambda' = \lambda - 2\pi t/T$$



Initial condition deformed into thin filaments that are transported Eastward by background zonal flow

Half way through simulation the deformational part of the flow reverses so that

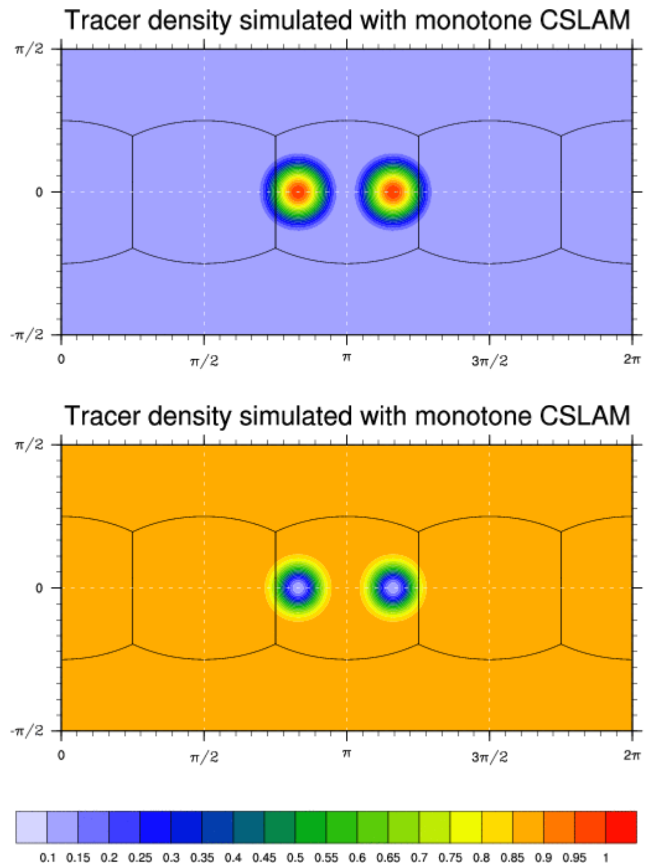
$$\psi(t=T) = \psi(t=0)$$

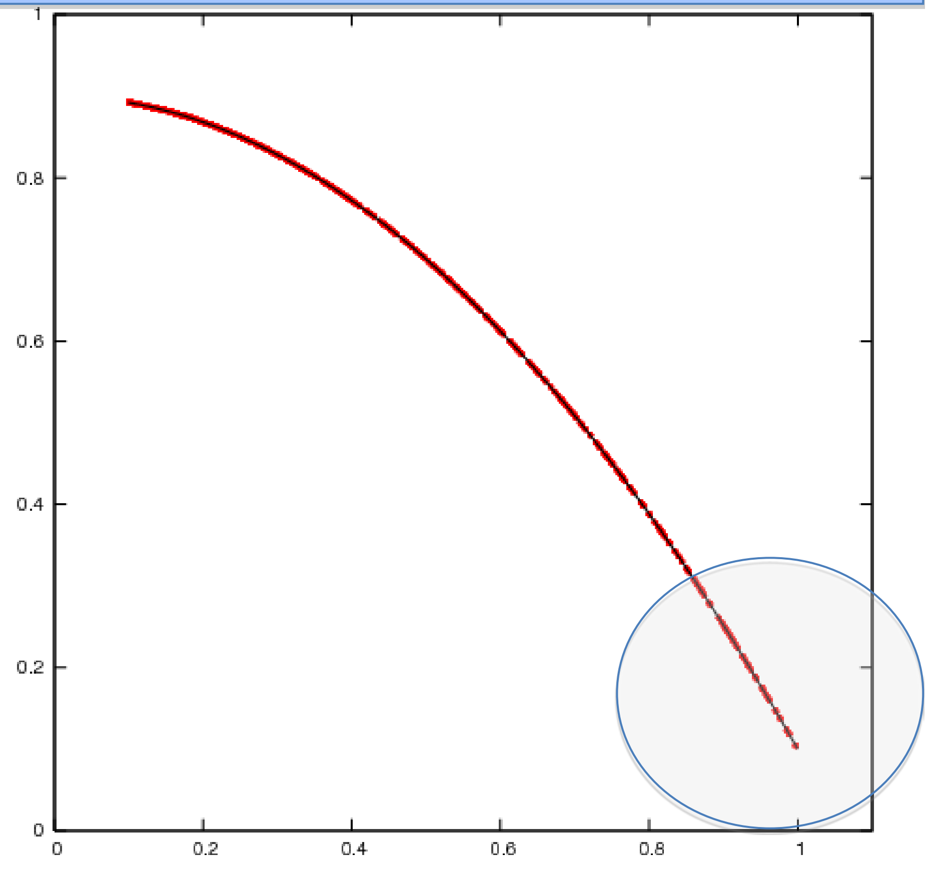
Convenient for computing conventional norms

Mixing diagnostics, however, computed at t=T/2

Preserving pre-existing functional relation between tracers under challenging flow conditions

Note: 1. **Max value decrease**, 2. Unmixing even if scheme is shape-preserving, 3. No expanding range unmixing





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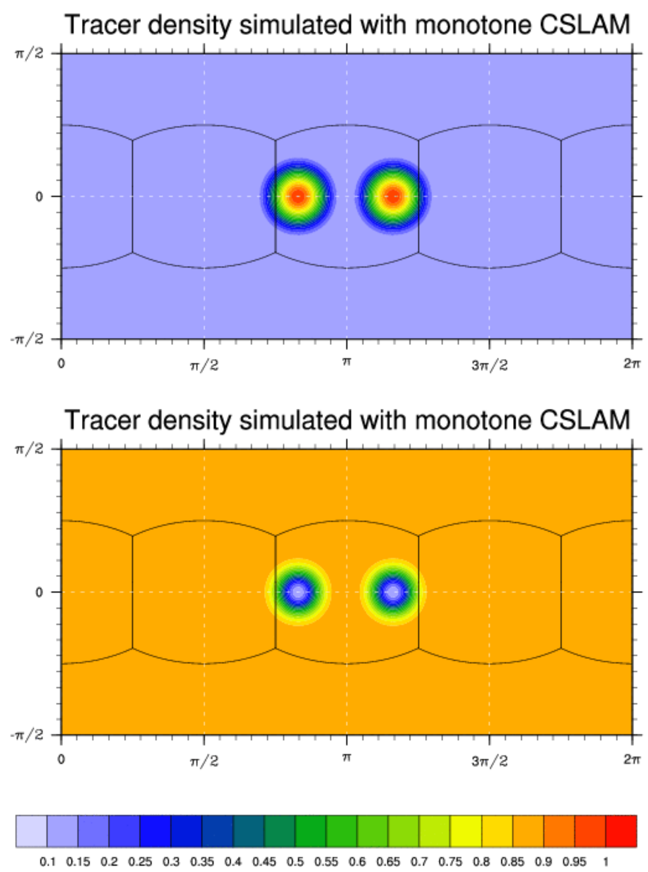

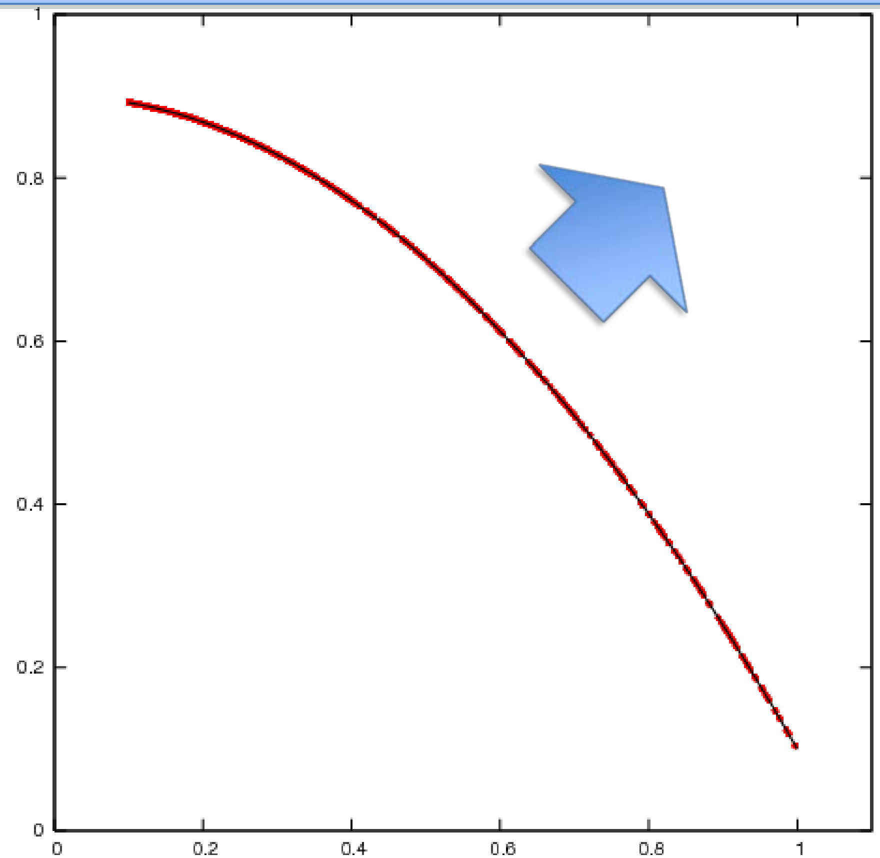

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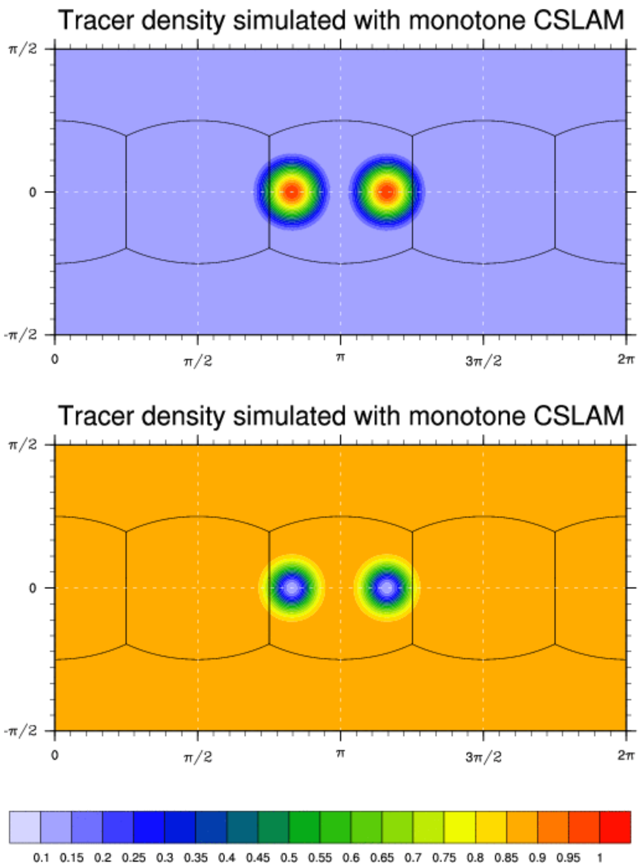
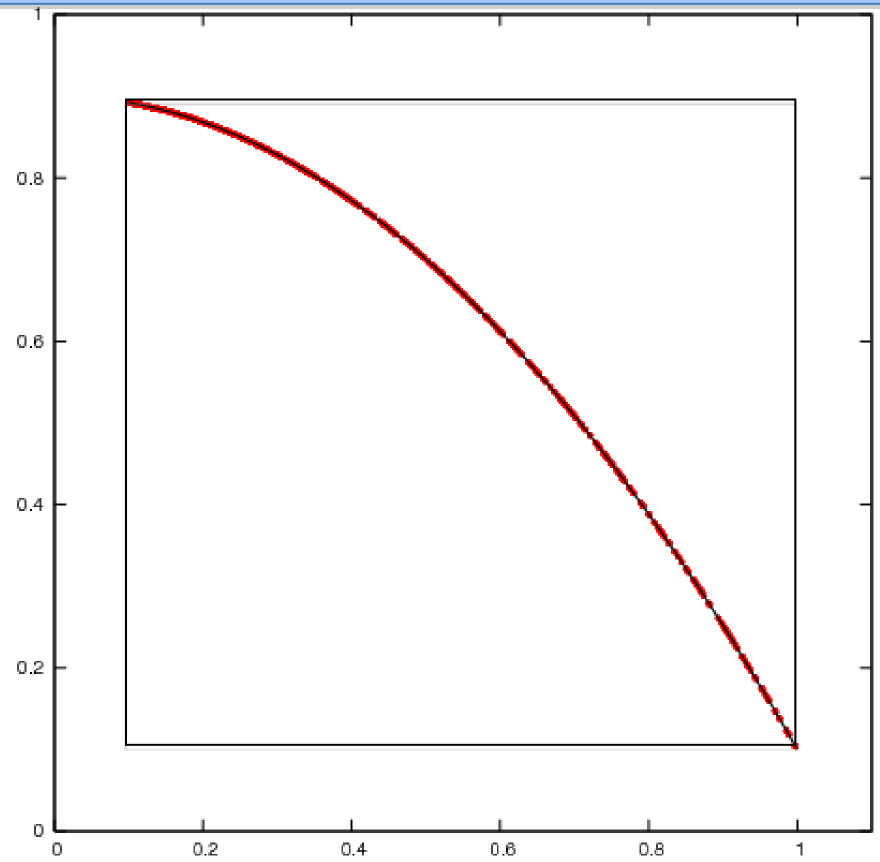
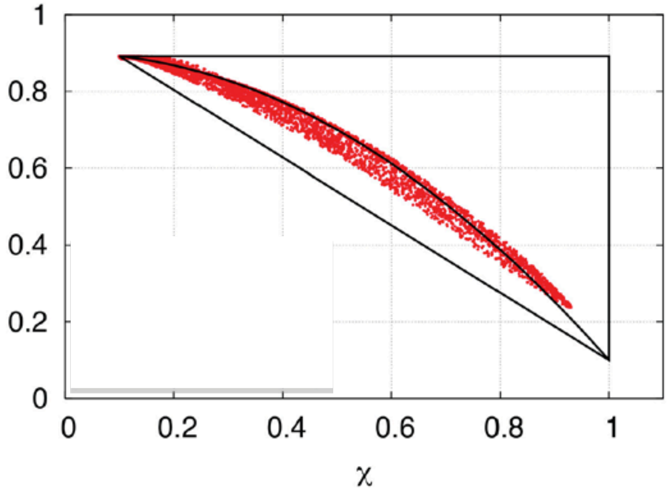
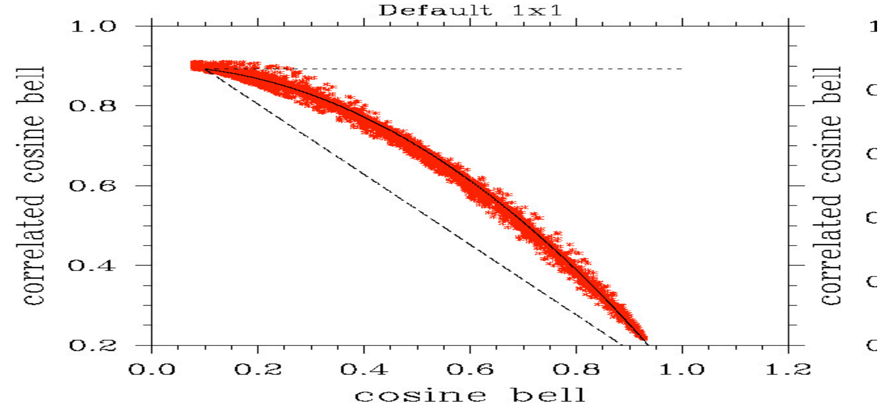




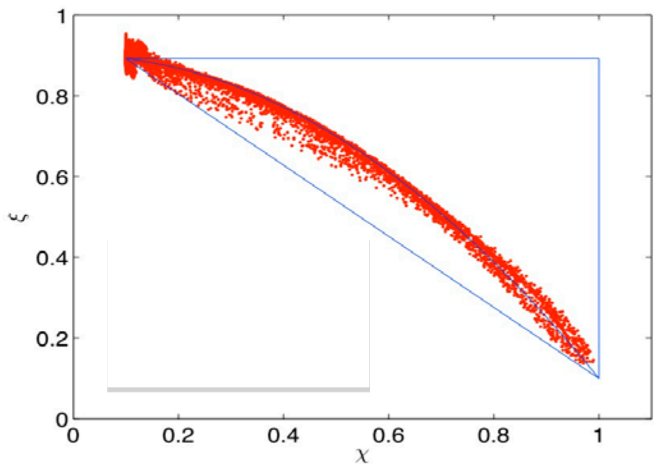
MPAS – 1.1 degree - CFL 0.7



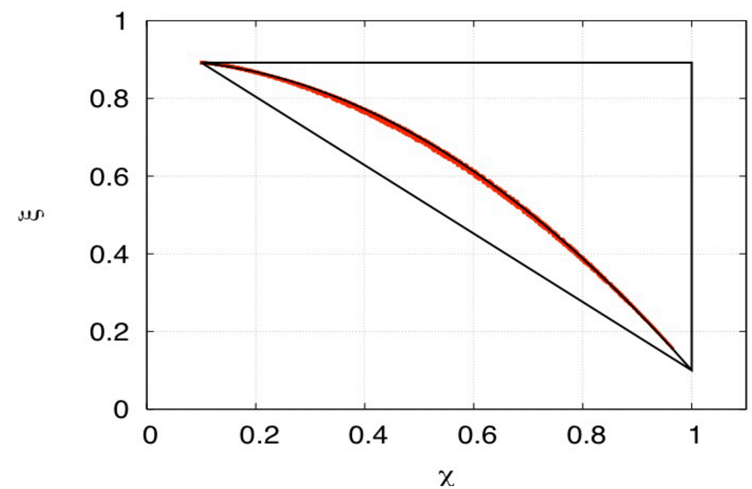
CAM-FV – 1 degree – CFL 1.6



CAM-SE – 1 degree – CFL 0.3



CSLAM – 1 degree – CFL 5.5



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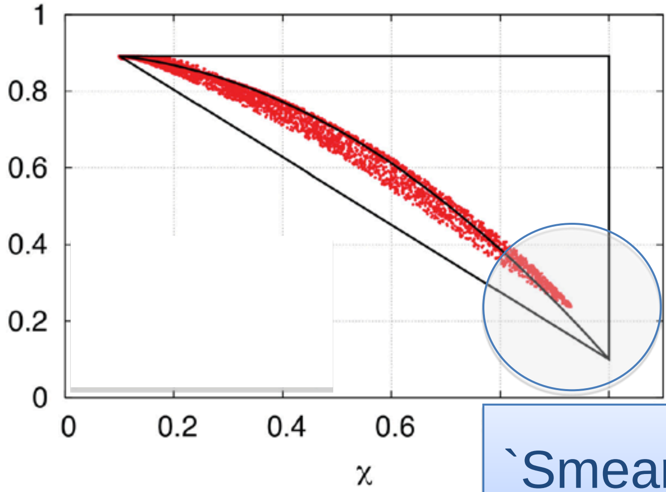

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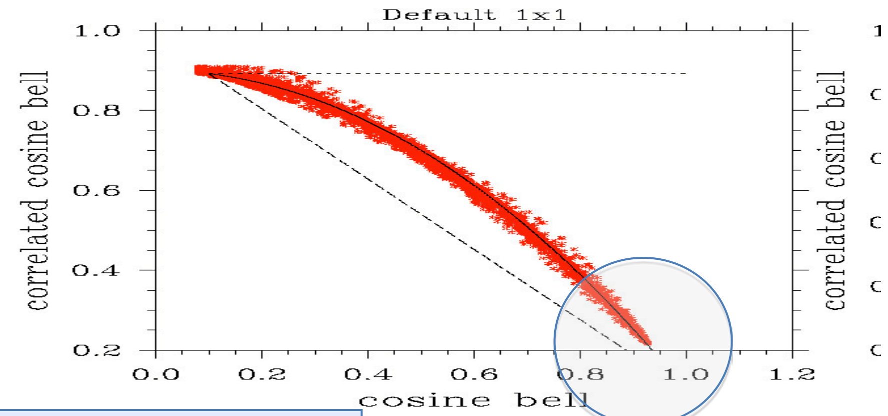
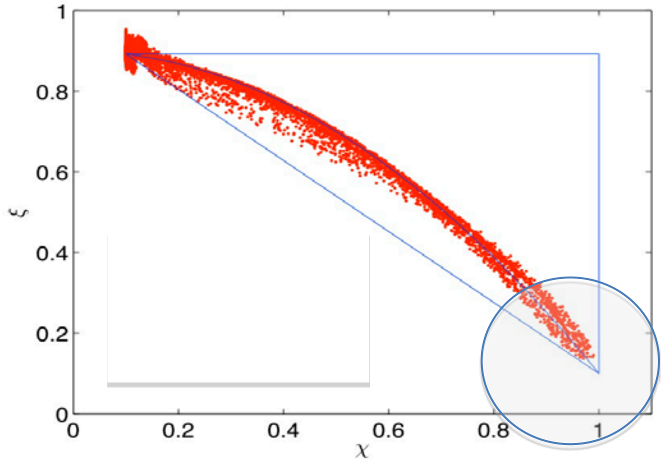


`Smearing' of extrema
(preservation of filaments)

CAM-SE – 1 degree – CFL 0.3



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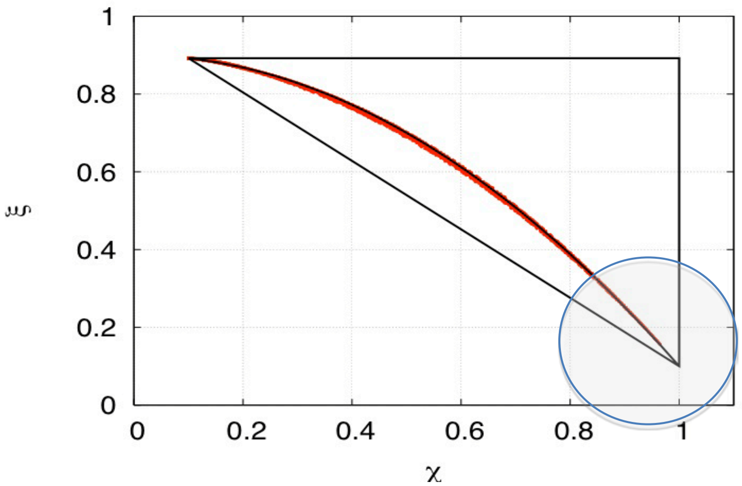

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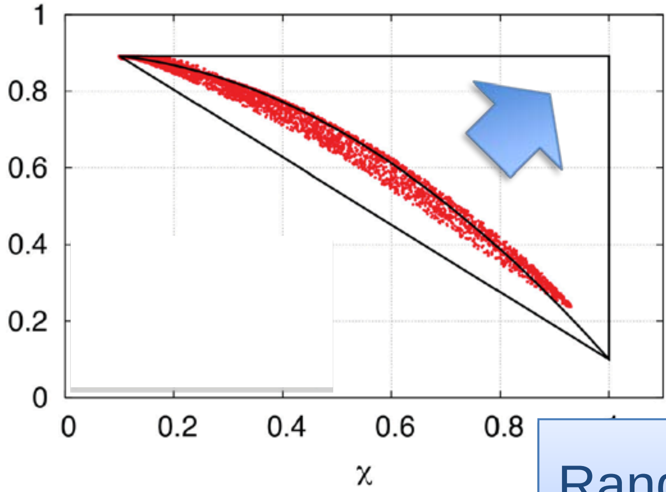

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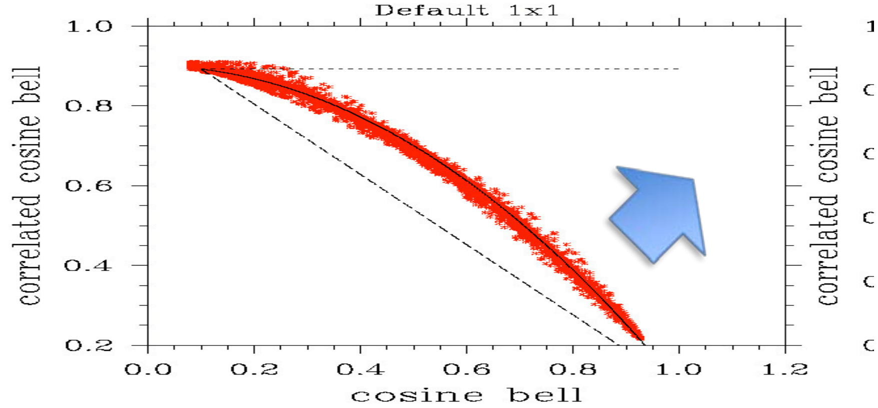
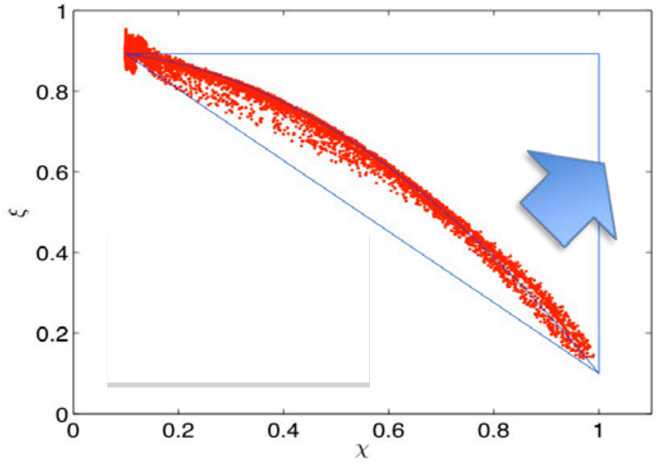
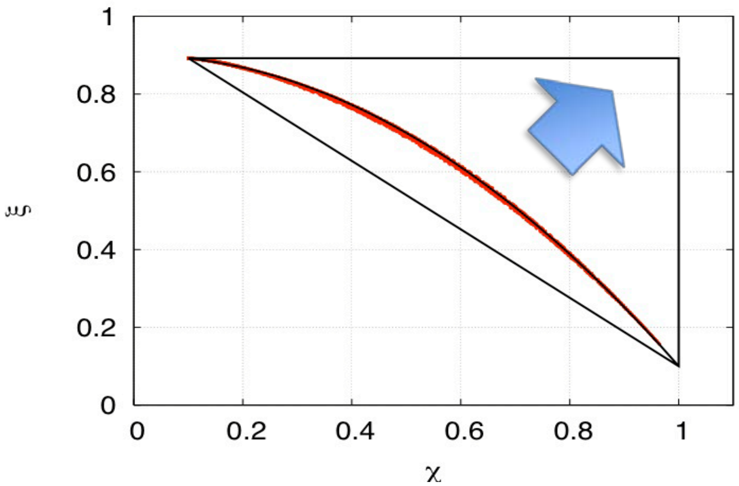


Range-preserving unmixing

CAM-SE – 1 degree – CFL 0.3



CSLAM – 1 degree – CFL 5.5



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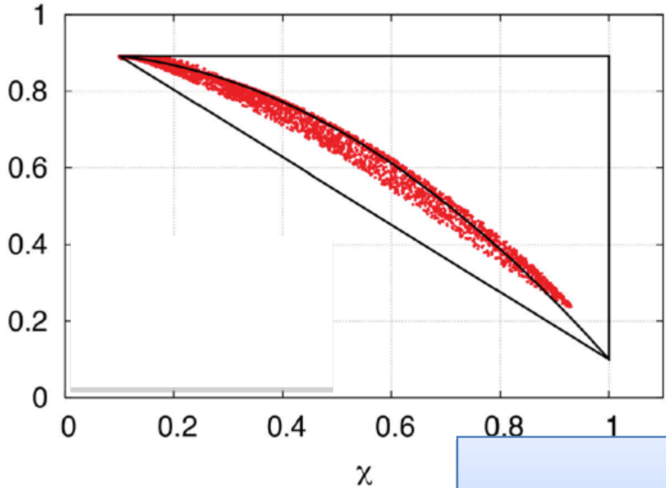

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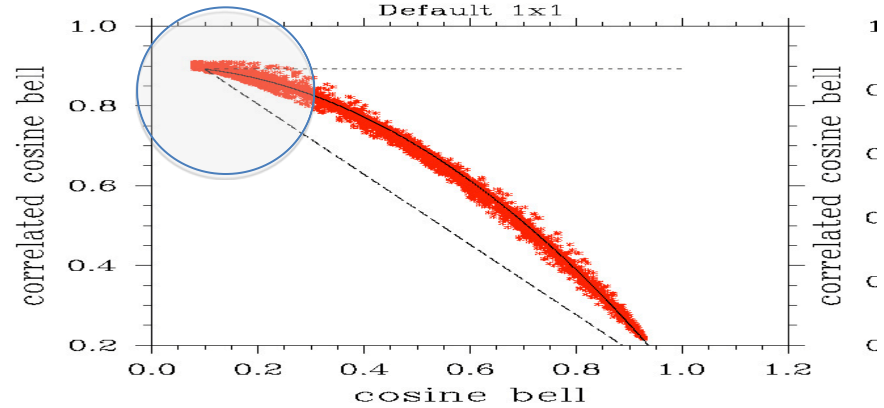
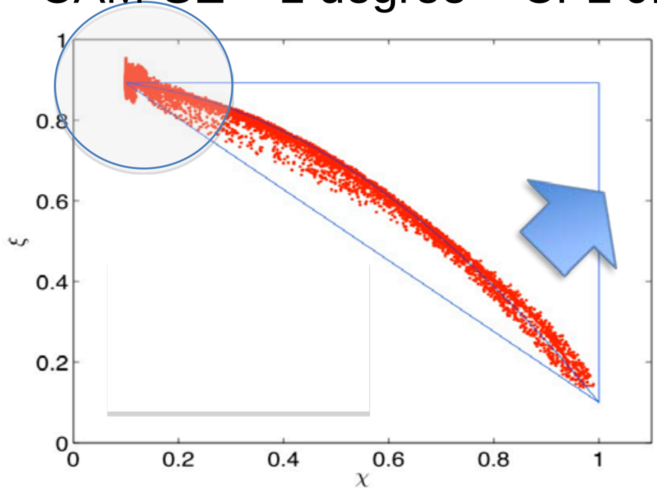


Expanding range unmixing

CAM-SE – 1 degree – CFL 0.3



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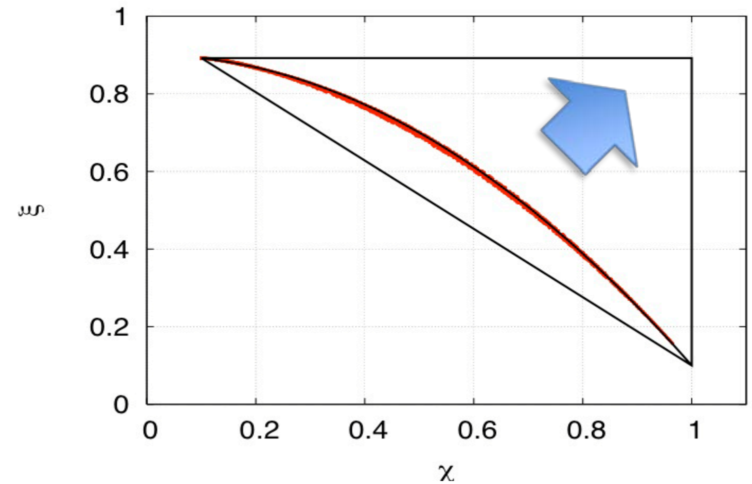

scheme	ℓ_1	ℓ_2	ℓ_∞	ℓ_r	ℓ_u	ℓ_o
1^{st}-order CSLAM	1.93×10^{-1}	3.82×10^{-1}	4.57×10^{-1}	6.02×10^{-3}	0.0	0.0
3^{rd}-order CSLAM	1.58×10^{-2}	3.28×10^{-2}	4.73×10^{-2}	7.55×10^{-4}	1.58×10^{-4}	3.79×10^{-4}
3^{rd}-order CSLAM with filter	1.58×10^{-2}	4.33×10^{-2}	8.91×10^{-2}	6.28×10^{-4}	6.73×10^{-5}	0.0

In terms of standard error norms the unfiltered version of CSLAM is more accurate than the filtered version

However, filtered version has less 'real mixing' and less spurious unmixing than unfiltered version

More physically motivated metric

If adding a simple non-linear reaction between two tracers it may be shown that the mixing diagnostics also predict amount of unphysical production/loss due to spurious mixing (see Lauritzen and Thuburn, 2011, for details).

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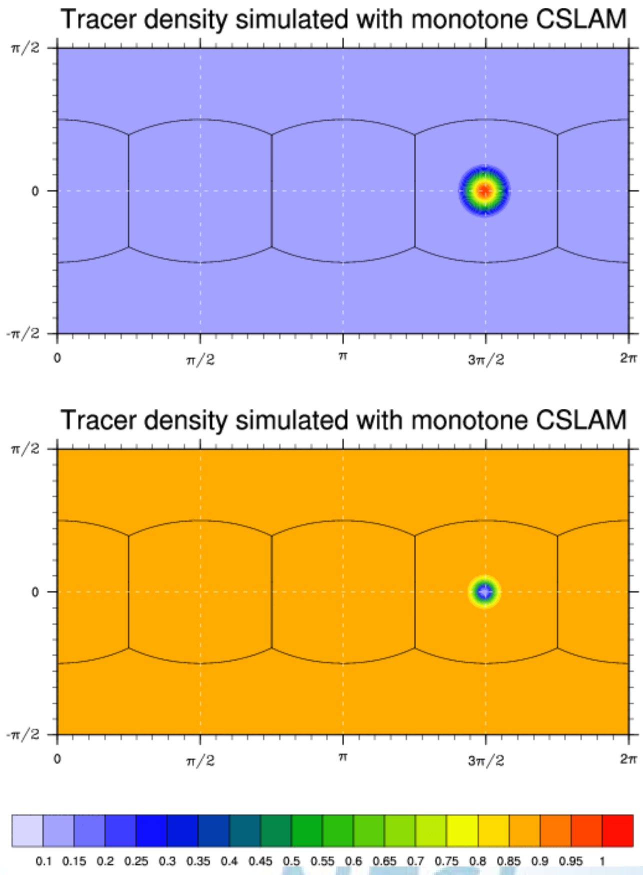
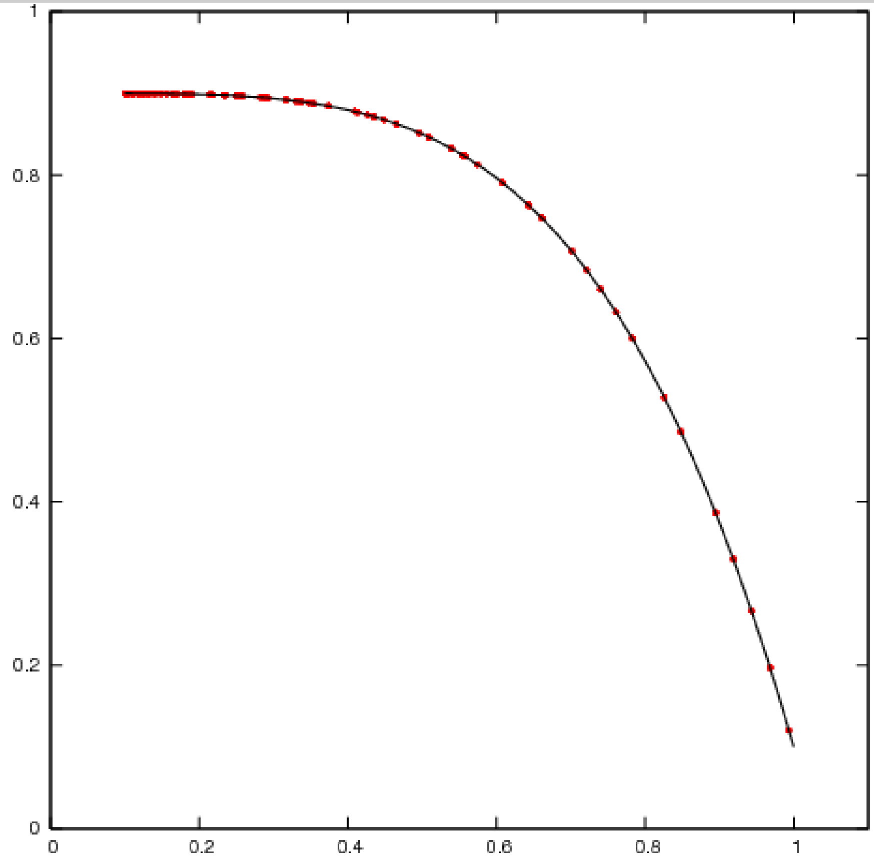
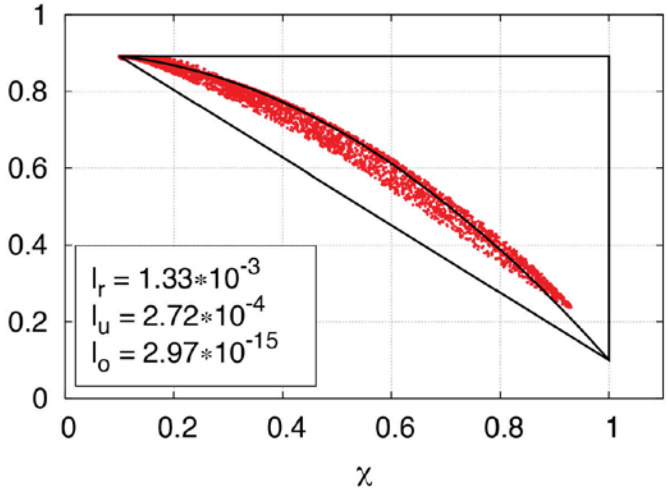
It is key that tracer features collapse to smaller scales (as in nature)



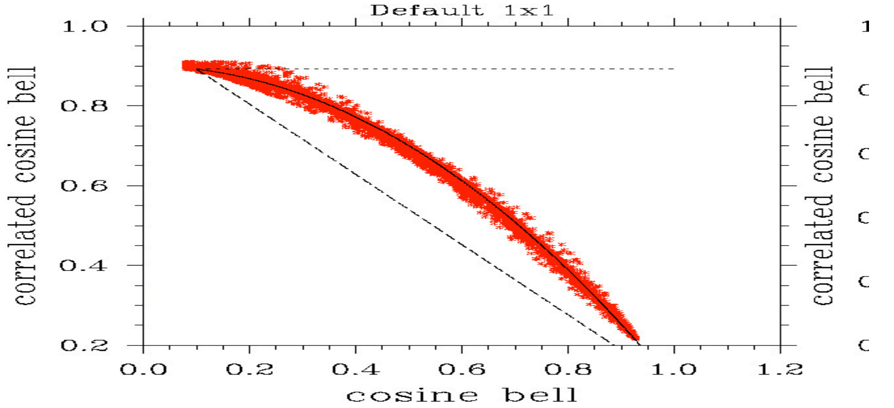


This setup uses a 4th-order non-linear relation $\Psi(\chi)=a\chi^4+b$

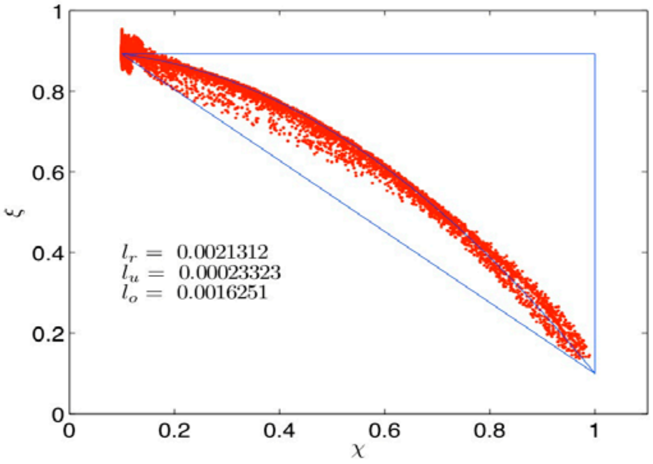
MPAS – 1.1 degree - CFL 0.7



CAM-FV – 1 degree



CAM-SE – 1 degree



CSLAM – 1 degree – CFL 1.4