# Estimation and Propagation of Errors in Ice Sheet Bed Elevation Measurements

Jesse V Johnson[1], Douglas Brinkerhoff[1], Sophie Nowicki[2], Kevin Sack[3]

1. University of Montana, 2. NASA Goddard, 3. University of Cape Town

14 February, 2012
Boulder, CO

# Problem Statement

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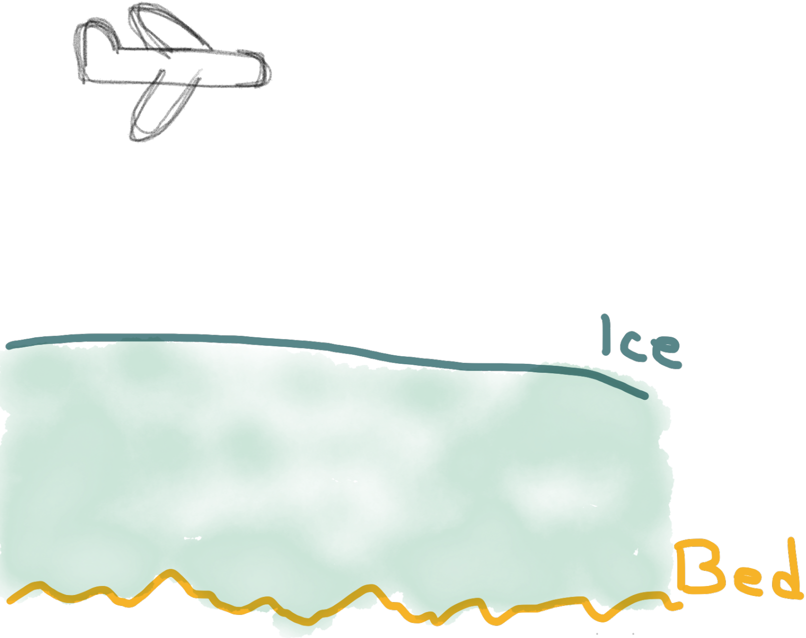

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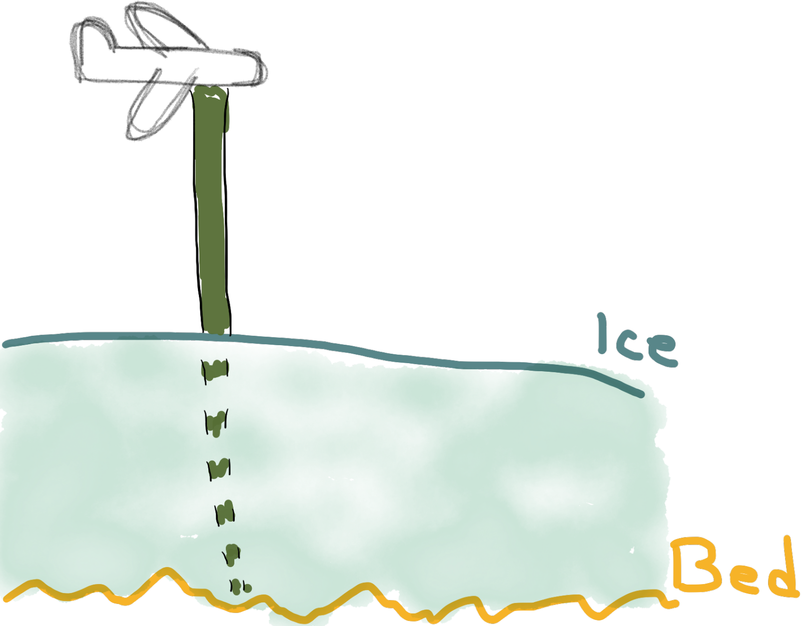

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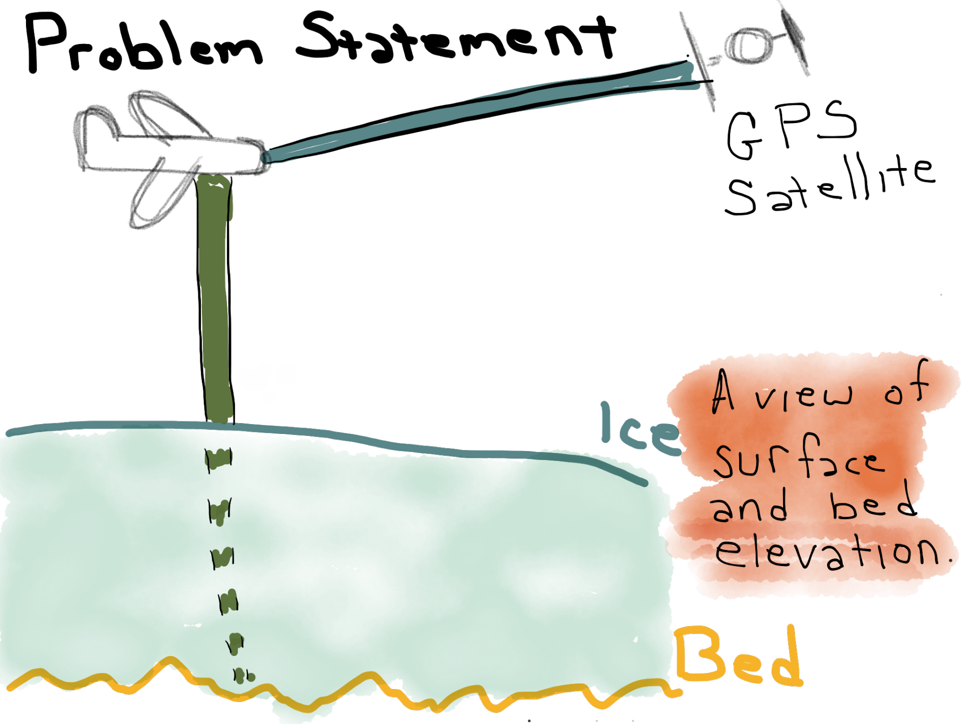

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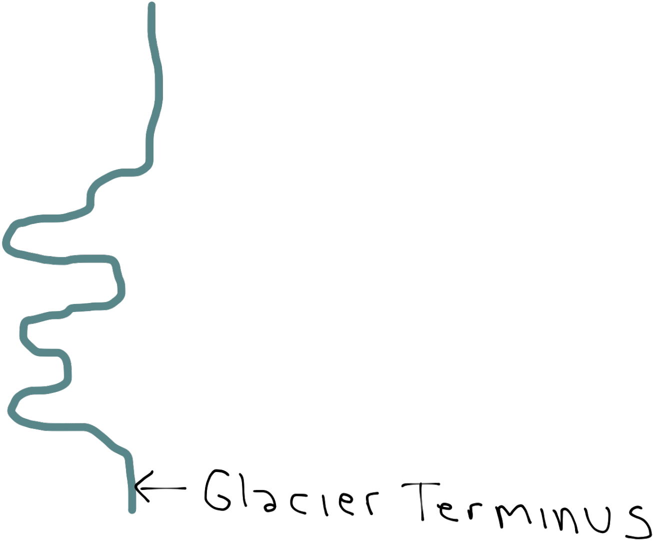
Plan View



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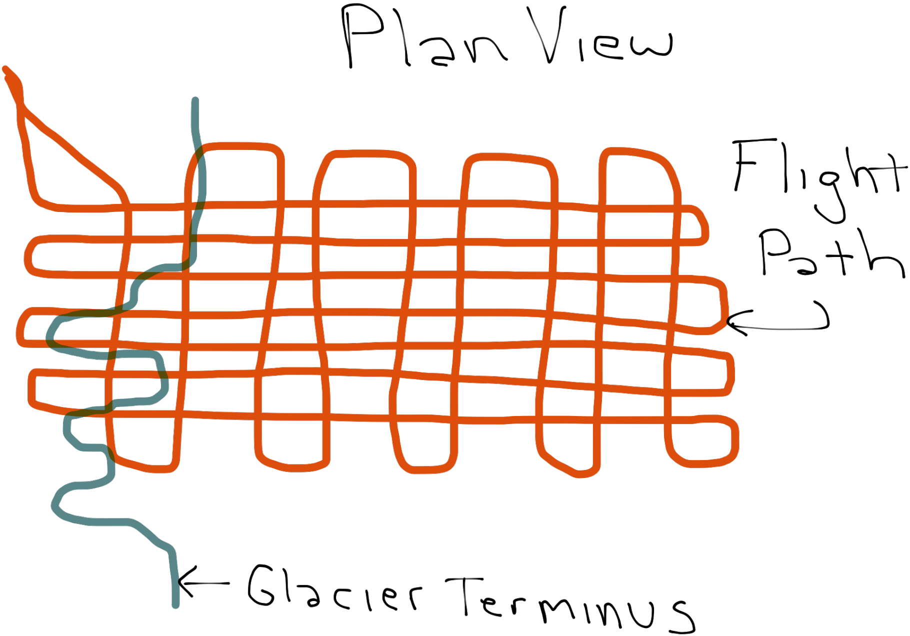

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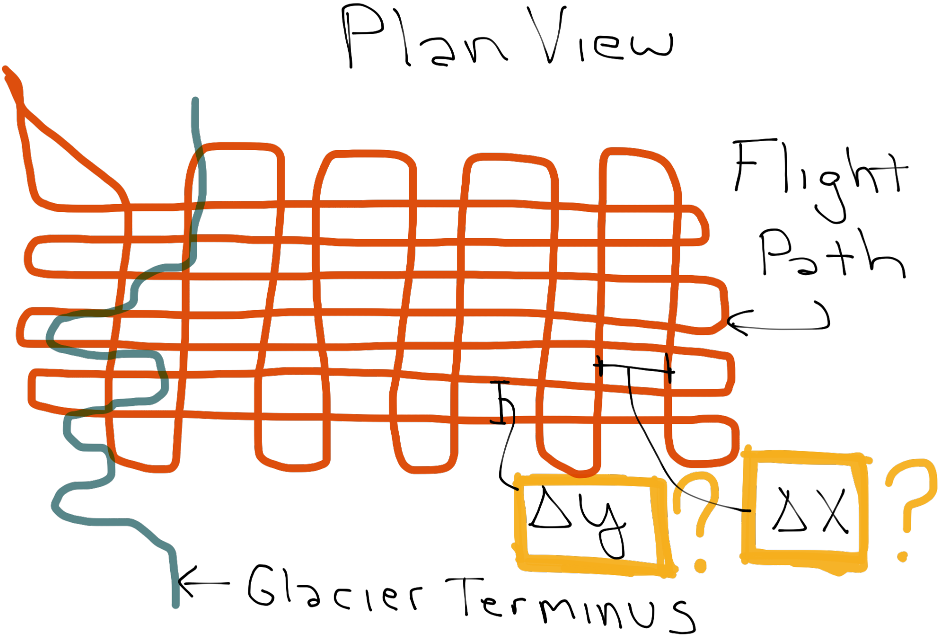

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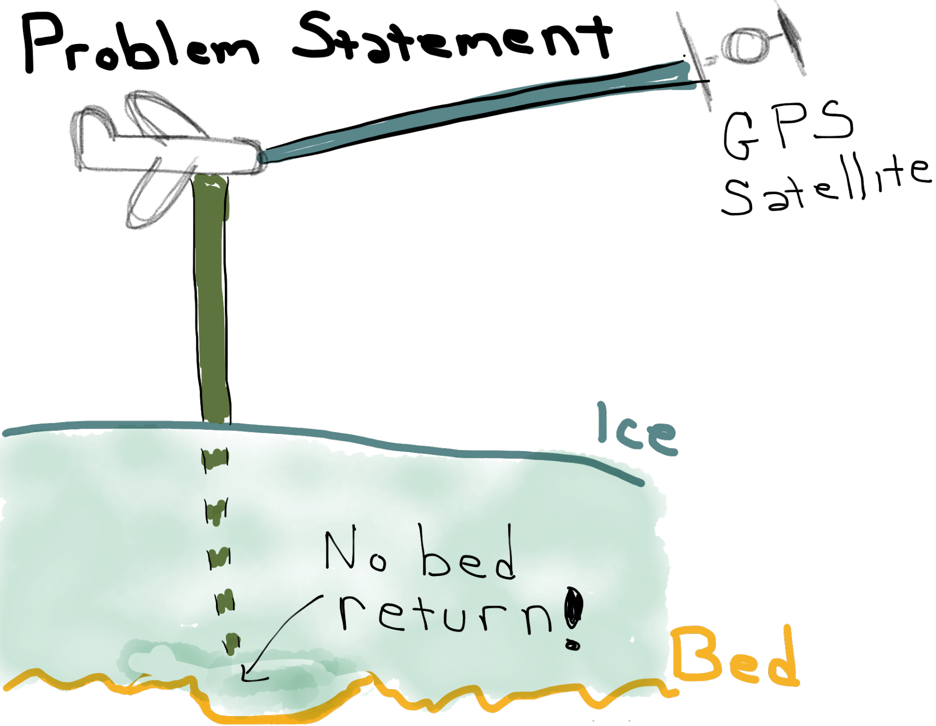

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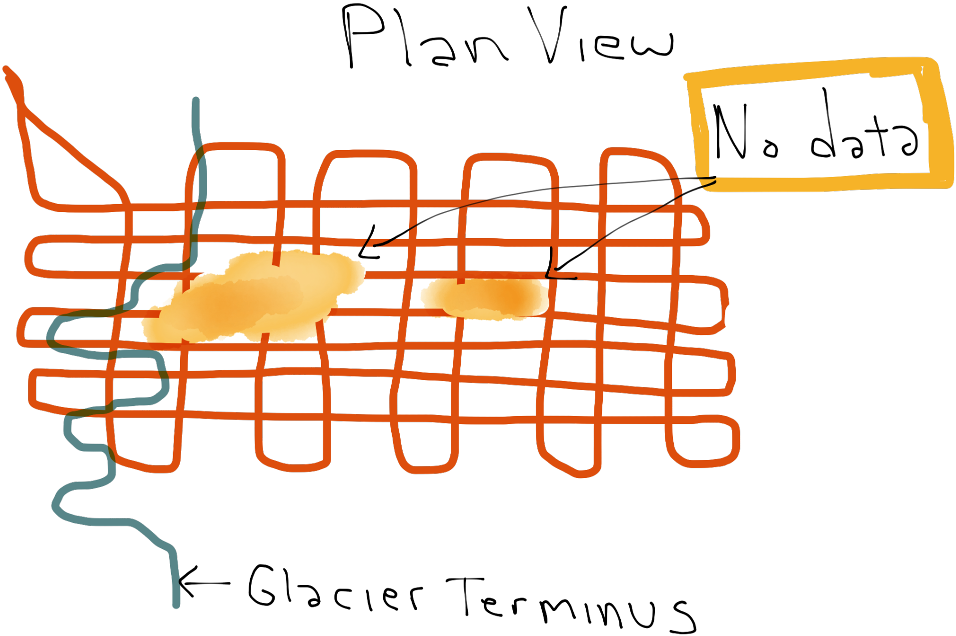

Goal: Estimate values between flightlines and in coverage gaps.

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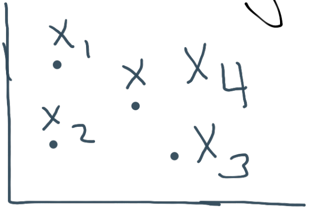
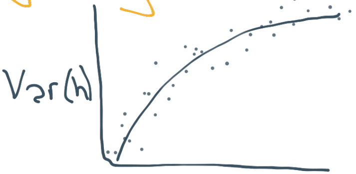



## Interpolation:

$$f(x) = \sum_{i \in \text{data}} w_i f(x_i)$$

## Inverse distance weighting

$$w_i = 1/d_i$$




## Kriging

$$w_i \Leftarrow \text{Variogram}$$

**Goal:** Estimate values between flightlines and in coverage gaps.

## Mass Conserving Bed

Continuity Equation:

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\vec{u} H) + \dot{a}$$

$H$ = thickness

$\vec{u}$ = velocity

$\dot{a}$ = accumulation/ablation

# Mass Conserving Bed

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\bar{u} H) + \dot{a}$$

observed

solved for

What about the flightlines, where H is known? Call this $H_0$

## Variational Form:

$$I = \int \left[ \frac{1}{2} \left( \nabla \cdot (\bar{u} H) - \dot{a} \right)^2 + \frac{\rho}{2} (H - H_0)^2 \right] d\Omega$$

$$\rho = \begin{cases} 1 & \text{on flight} \\ 0 & \text{off flight} \end{cases}$$

# Mass Conserving Bed

observed

$$\frac{\partial H}{\partial t} = -\nabla \cdot (\bar{U} H) + \dot{a}$$

solved for

What about the flightlines, where H is known? Call this $H_o$

## Variational Form:

$$I = \int \left[ \frac{1}{2} \left( \nabla \cdot (\bar{U} H) - \dot{a} \right)^2 + \frac{p}{2} (H - H_o)^2 \right] d\Omega$$

$$p = \begin{cases} 1 & \text{on flight} \\ 0 & \text{off flight} \end{cases}$$

Minimize $I$

Use 1st variation: $\delta I(H)(\delta H)$

# Objective:

Find errors in both Krigging and mass conserving beds.

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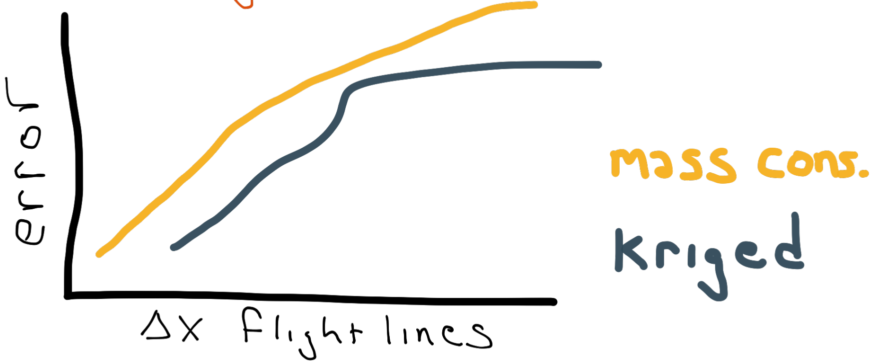
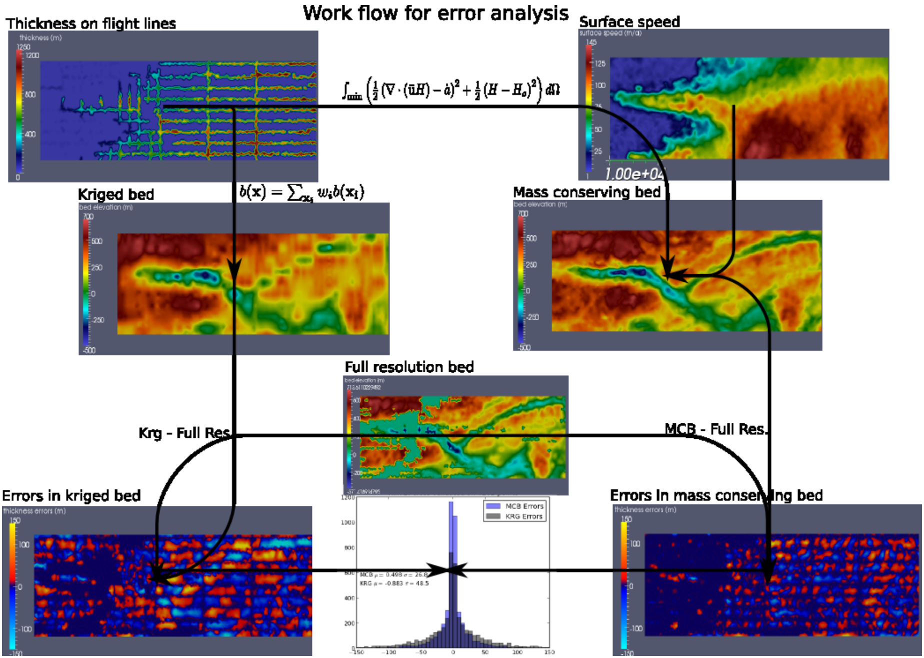
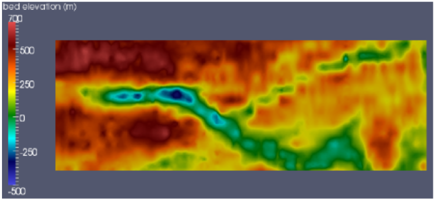
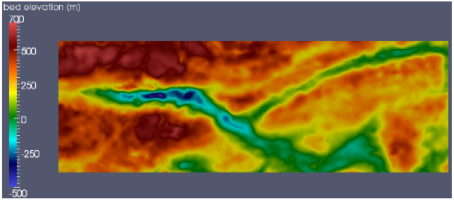



Work flow for error analysis
Thickness on flight lines
thickness (m)
Surface speed
surface speed (m/a)
$\int_{\min}\left(\frac{1}{2}(\nabla\cdot(\bar{u}H)-\dot{a})^2+\frac{1}{2}(H-H_o)^2\right)d\Omega$
1.00e+04
Kriged bed
bed elevation (m)
$b(\mathbf{x})=\sum_{\mathbf{x}_i} w_i b(\mathbf{x}_i)$
Mass conserving bed
bed elevation (m)
Full resolution bed
bed elevation (m)
Krg - Full Res.
MCB - Full Res.
Errors in kriged bed
thickness errors (m)
Errors in mass conserving bed
thickness errors (m)
MCB Errors
KRG Errors
MCB $\mu$ = 0.498 $\sigma$ = 26.8
KRG $\mu$ = -0.883 $\sigma$ = 48.5

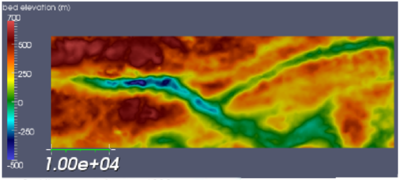
## Kriged bed at 1.0 km spacing

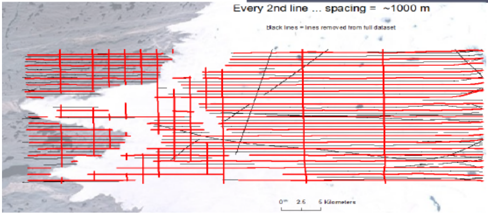



## Mass conserving bed at 1.0 km spacing

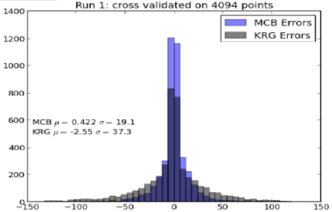



## Mass conserving bed at 500 m spacing

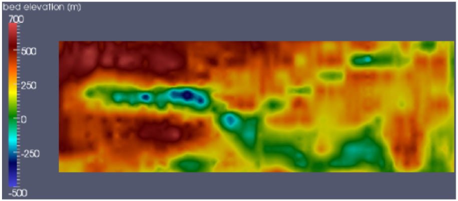



## Error Structure



## Kriged bed at 1.5 km spacing

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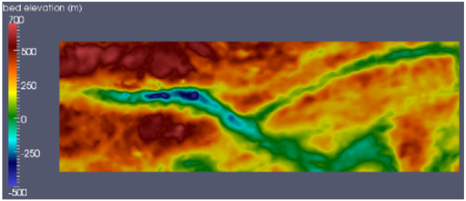
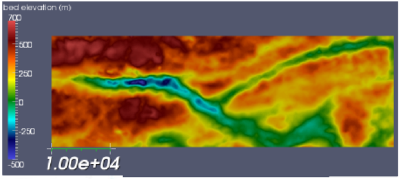
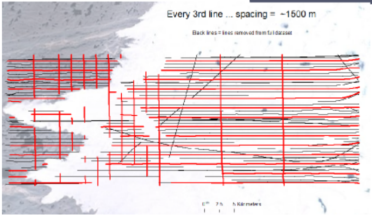



## Mass conserving bed at 500 m spacing

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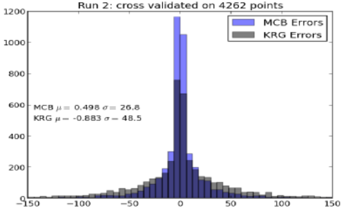
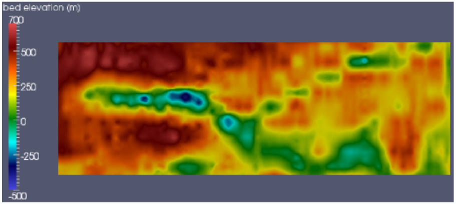


## Kriged bed at 2.0 km spacing

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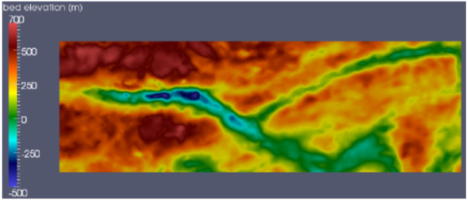
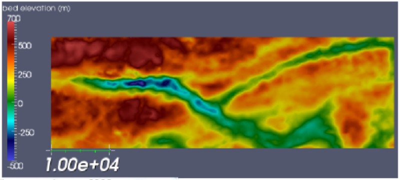
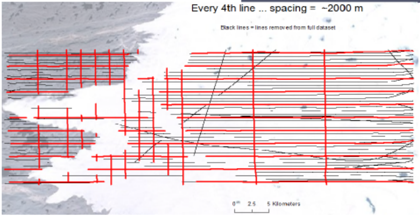



## Mass conserving bed at 500 m spacing

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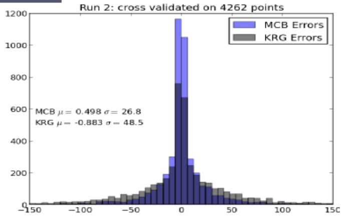

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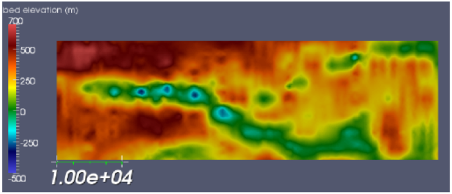


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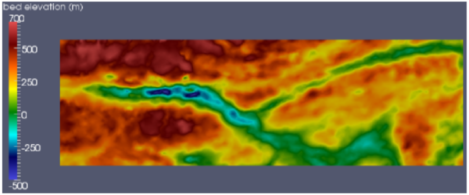


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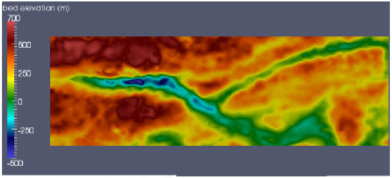


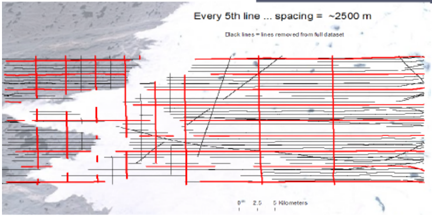


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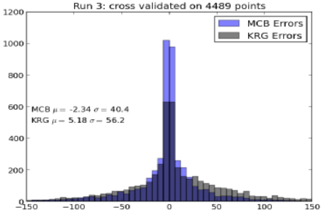
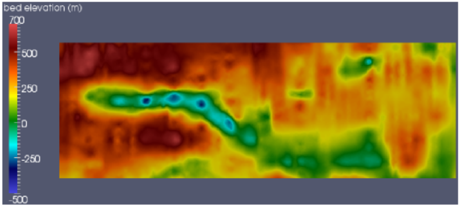
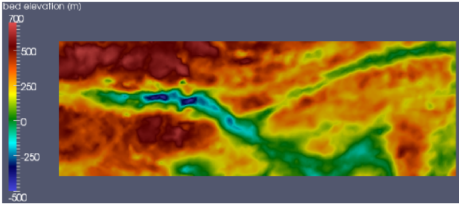


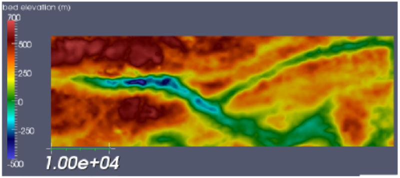
## Kriged bed at 3.0 km spacing

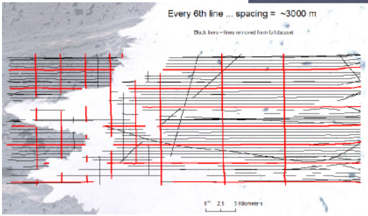



## Mass conserving bed at 3.0 km spacing




## Mass conserving bed at 500 m spacing

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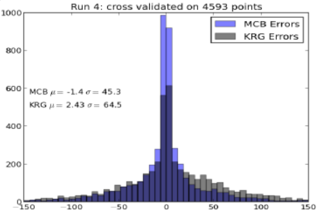
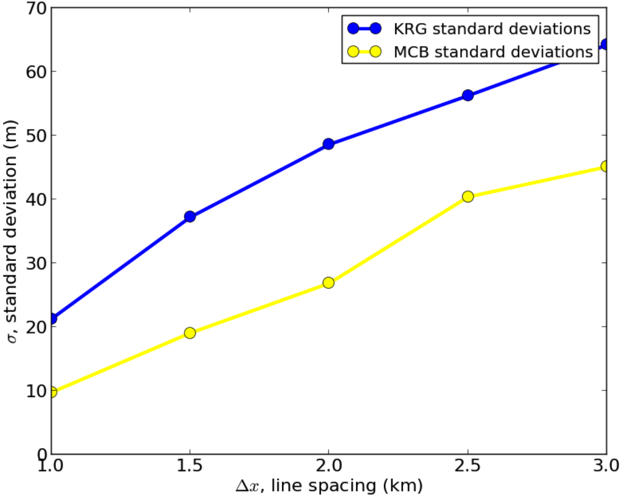
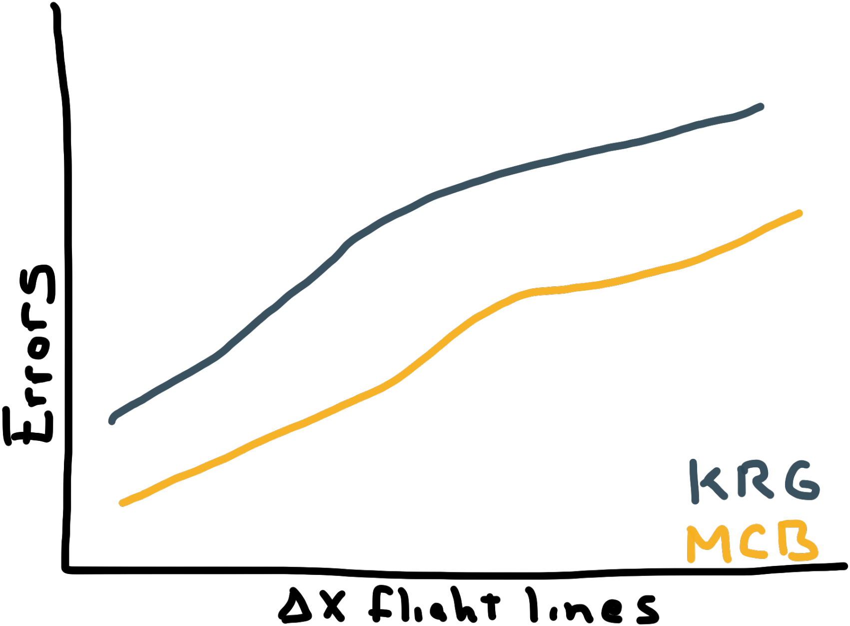


# Errors in the bed vs. flight spacing



# How much do bed errors matter?



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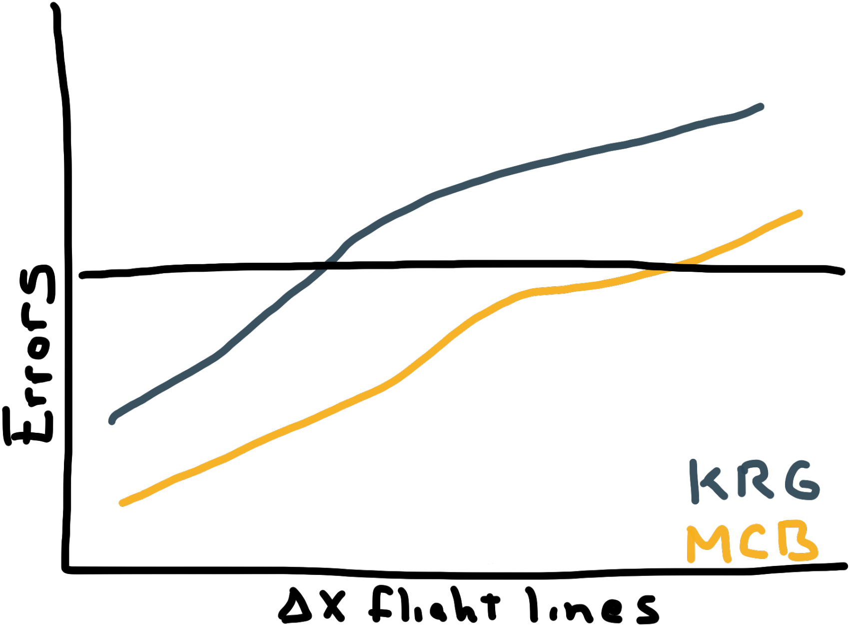

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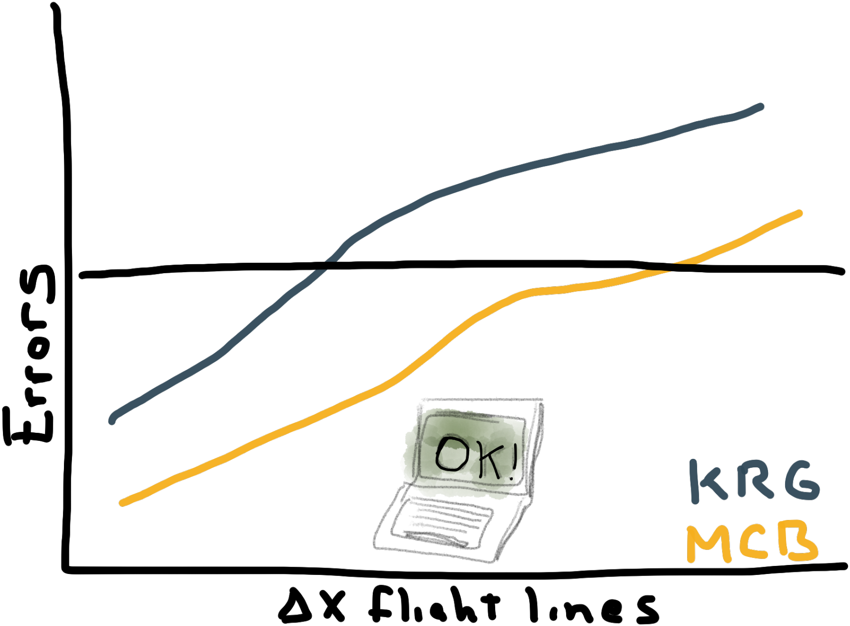

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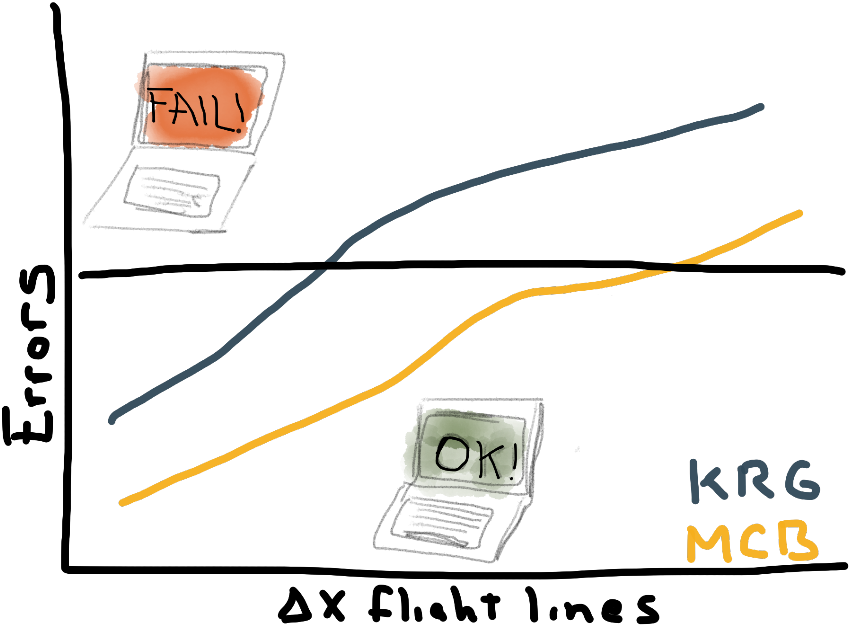

# Numerical Experiment

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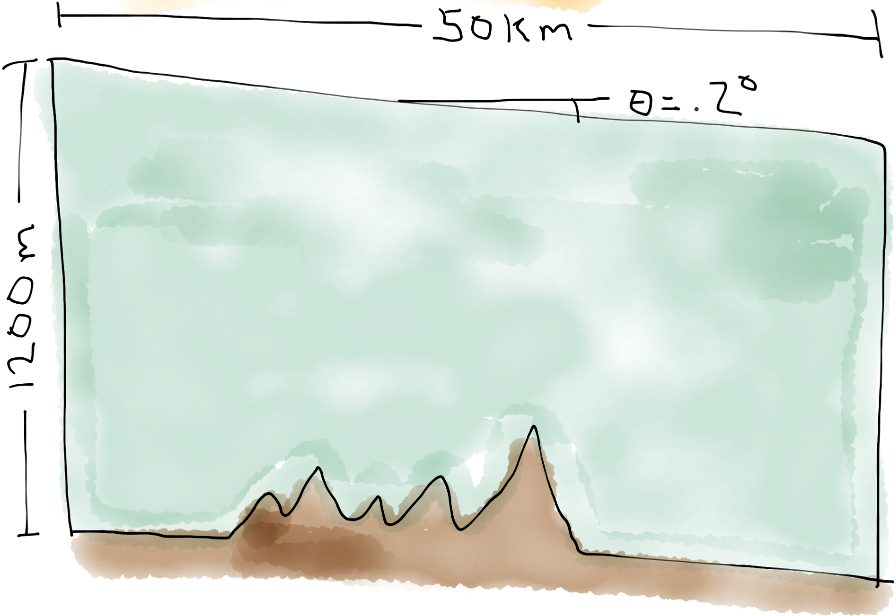

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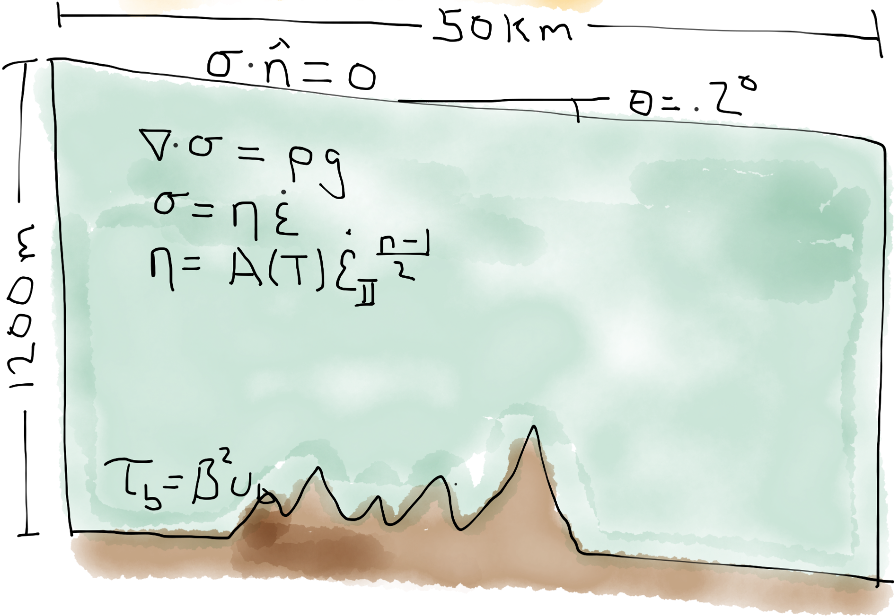

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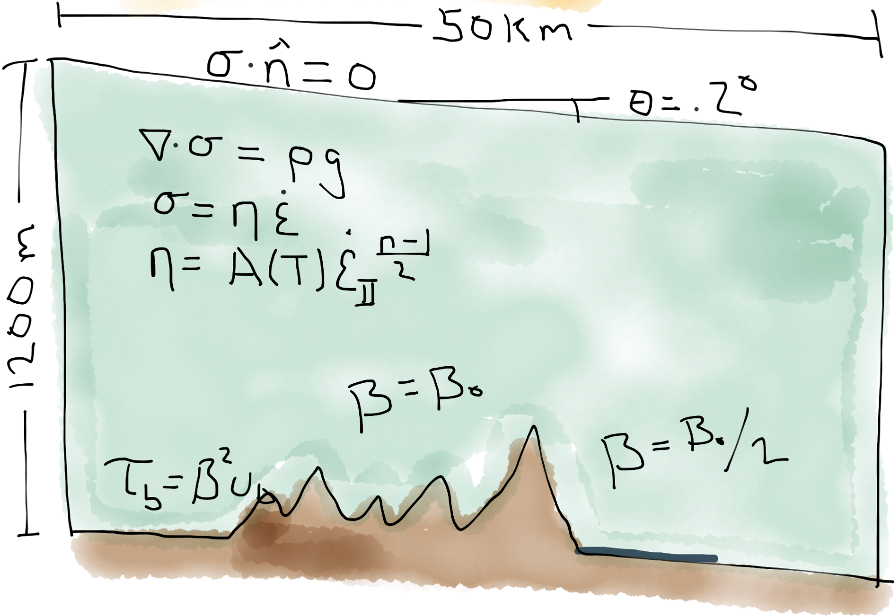

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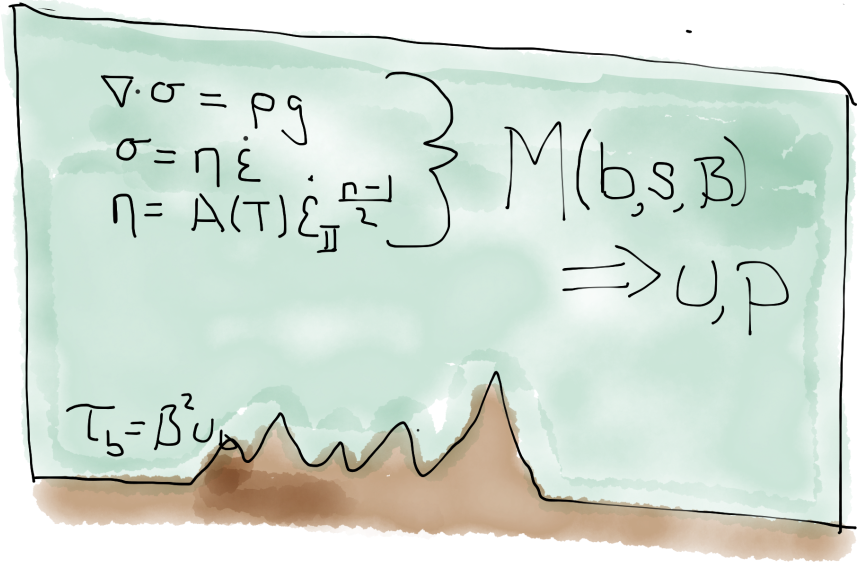

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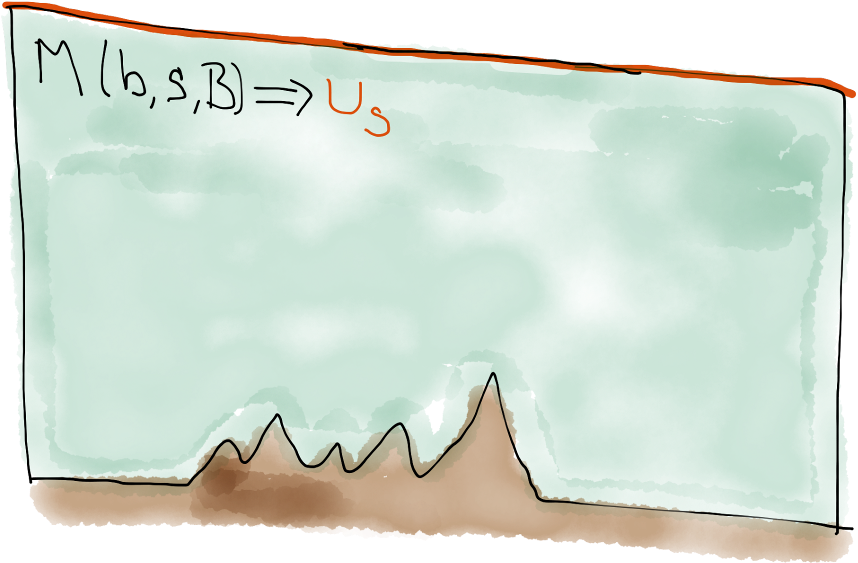

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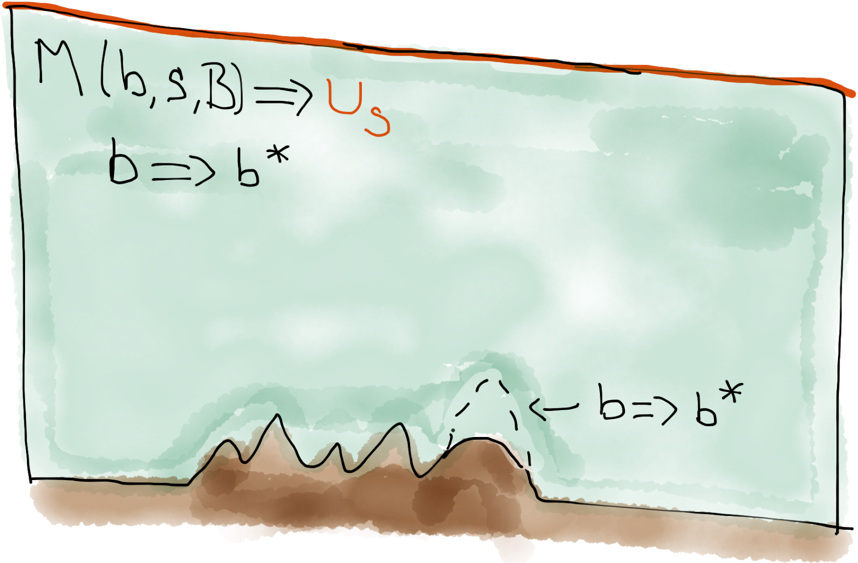

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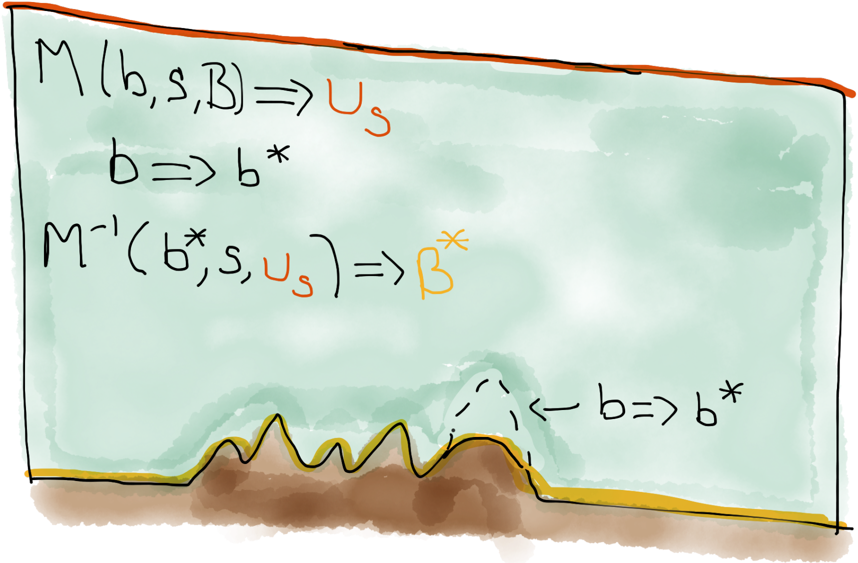

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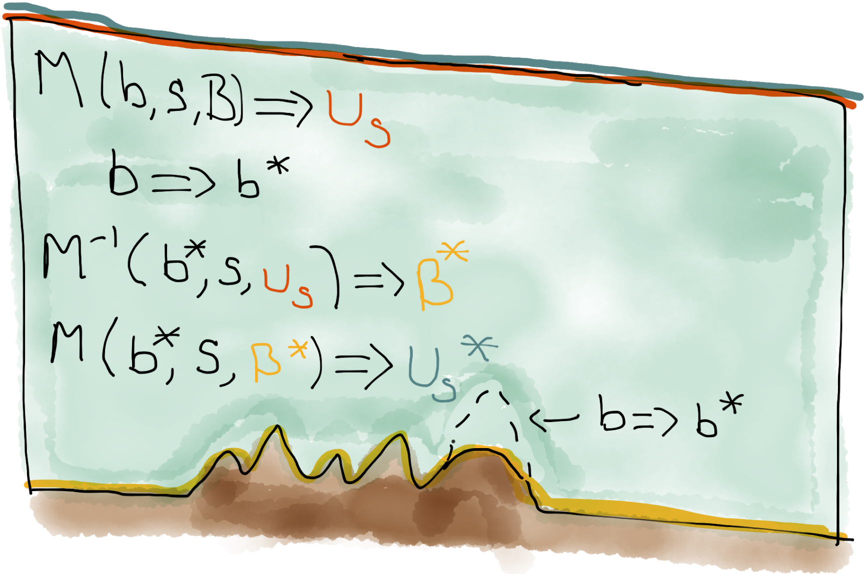

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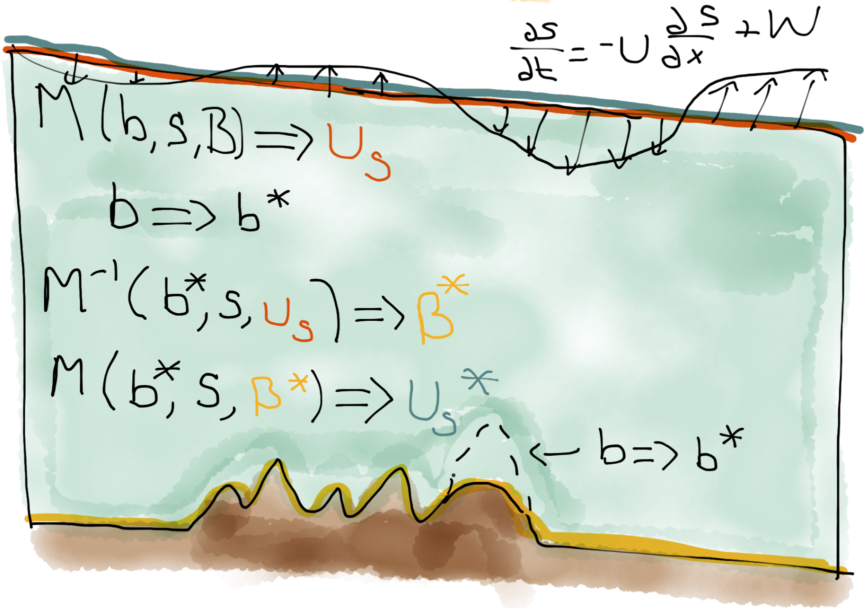
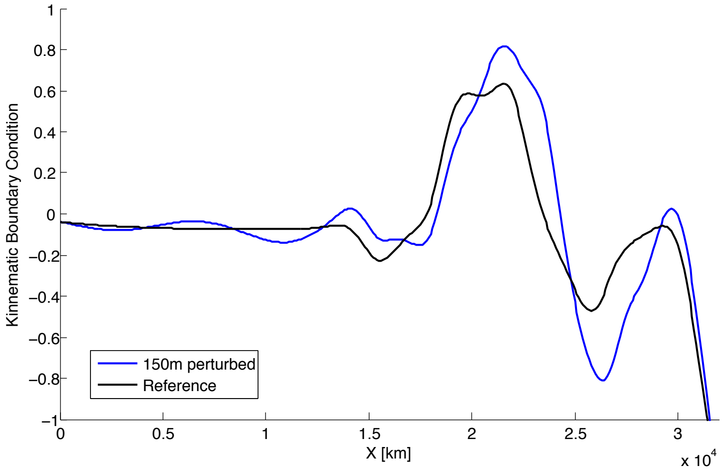


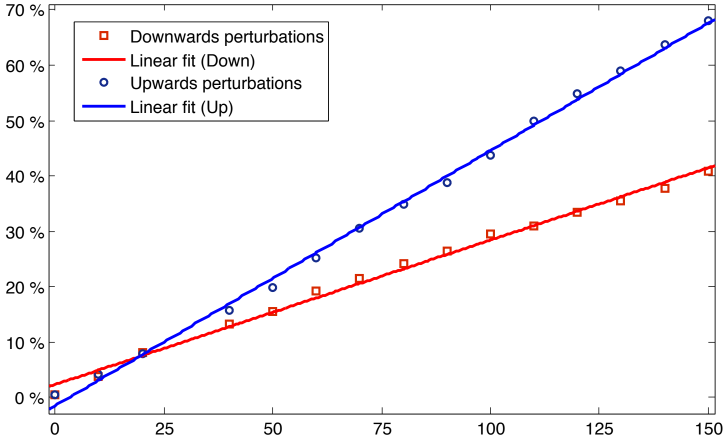
# Errors in the kinematic boundary condition

Mathematically, this is the surface rate of change, given by $\frac{\partial S}{\partial t} = -u\frac{\partial S}{\partial x} + w$



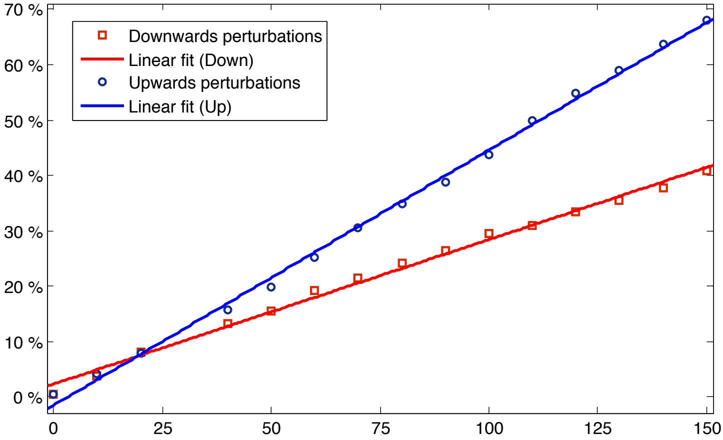
# Total error in volume rate of change:

Or, $\frac{\partial \Delta V}{\delta t} = \int |\frac{\partial S}{\partial t} - \frac{\partial S^*}{\partial t}| dx / \int \frac{\partial S}{\partial t} dx$



# Limit volume change errors to 10%

## Perturbations can be about 25 m



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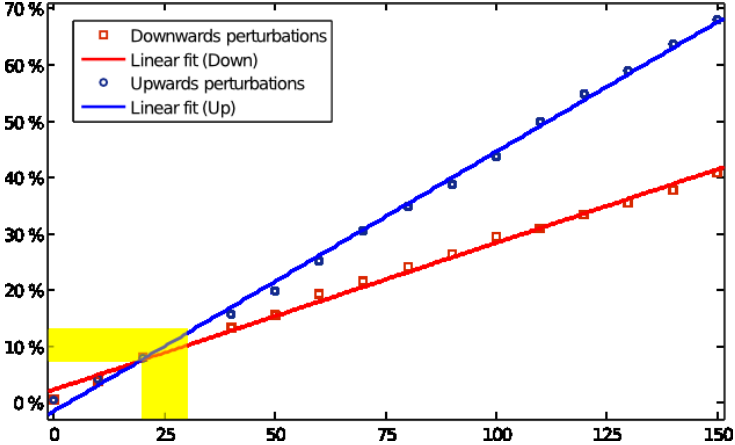
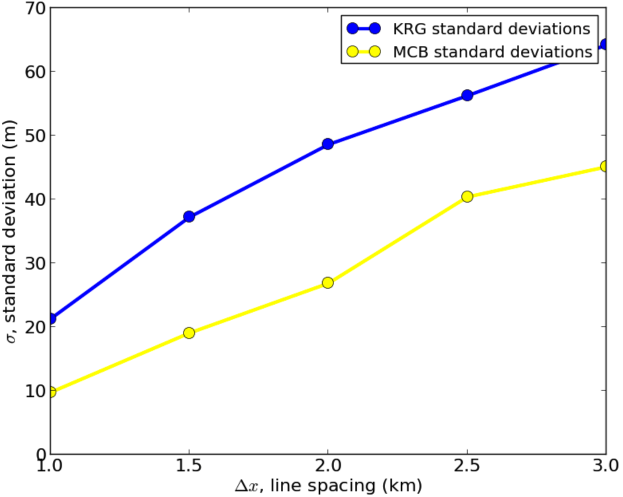


# 25 m errors in bed can be related to flight line spacing

Although $1\sigma$ only contains $\sim$ 2/3 of errors.



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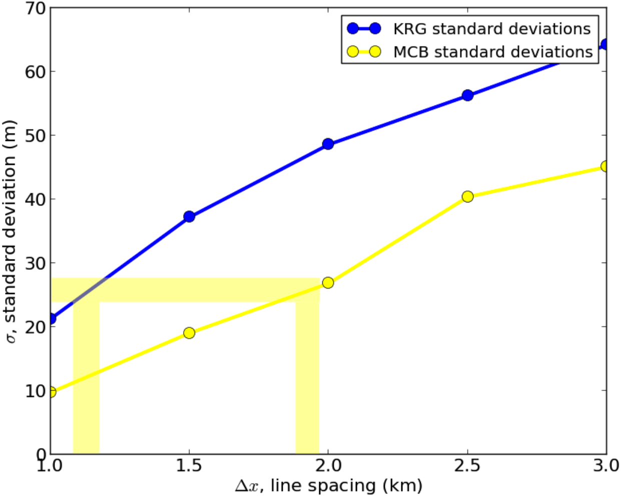
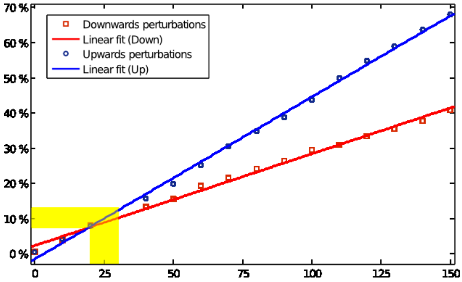
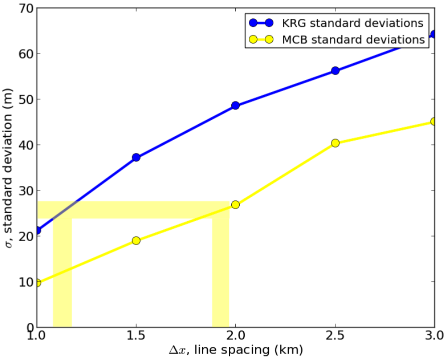


# Conclusions

Flight line spacing should be $\sim$ 1.8 km if MCB is used.



# Conclusions

This is the first example in a general framework for "*physics based interpolation*"

## Our variational problem

$$\mathcal{L}(H, \bar{\mathbf{u}}, \dot{a}) = \int \frac{1}{2} \underbrace{\rho (H - H_o)^2}_{\text{the data}} + \frac{\gamma}{2} \underbrace{(\nabla \cdot (\bar{\mathbf{u}} H) - \dot{a})^2}_{\text{PDE involving data}} d\Omega$$

$$\delta \mathcal{L}(\delta H, \bar{\mathbf{u}}, \dot{a}) = 0$$