Ill-Posed Glacier Volume Estimation

AKA: Please Don't Shoot the Messenger

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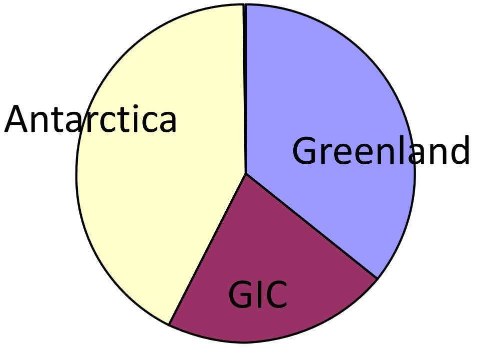
Who Gives A Rat's Derriere About GIC Volume?

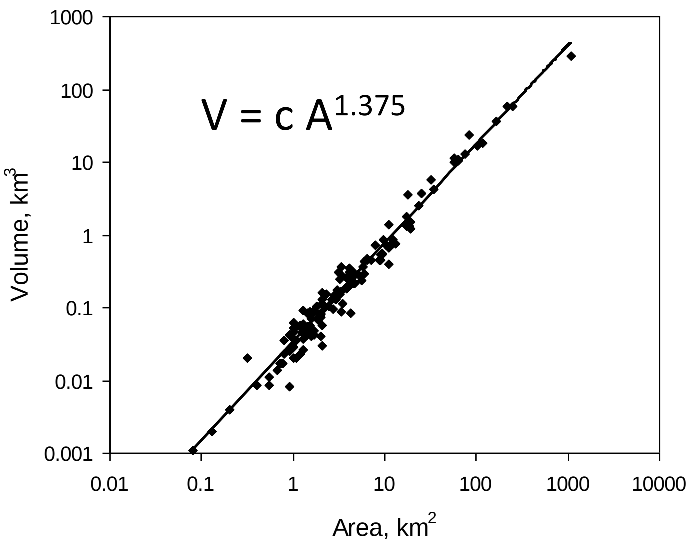
- 200,000+ glaciers and ice caps (GIC), but we only have measured volumes for about 100. Ouch.
- GIC will contribute about 1/3 of SLR over next 100 years.
- GIC are a leading term in today's sea-level rise.
- Can't predict their contribution to SLR if we don't know their volume.



Approx. SLR contributions from ice over next 100 years

So How Do We Calculate Volume?

- Volume-Area Scaling
  - Cute, simple, direct.
  - Imprecise? (Stay tuned!)



- Numerical Inversions
  - Complex but versatile.
  - Hard to model 200,000+ glaciers, but it has been done!
    - Huss and Farinotti, 2012
  - Presumably more precise?

$$h_i = \sqrt[n+2]{\frac{(1-f_{\mathrm{sl}})\cdot q_i}{2A_{\mathrm{f}}(T)}\cdot\frac{n+2}{(F_{\mathrm{s,i}}\rho g \sin\overline{\alpha_i})^n}},$$

Which Approach is More Accurate?

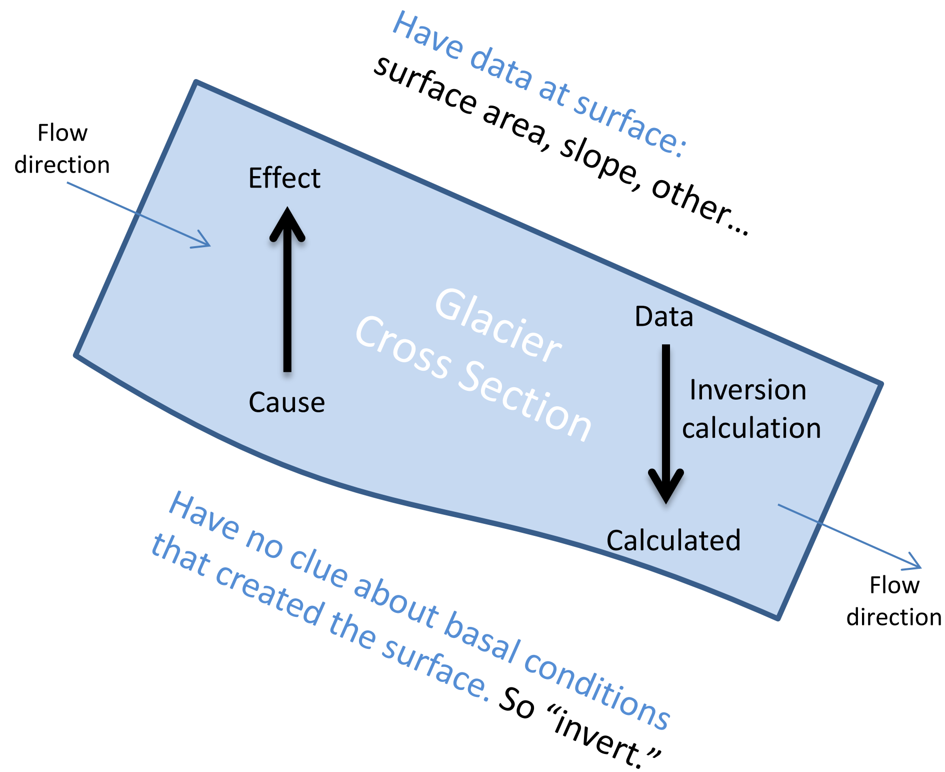
Modeling Versus Scaling

- Neither.
  - (Please don't shoot the messenger.)

- Both.
  - They give darn near the exact same accuracy and precision.
  - Can prove it.

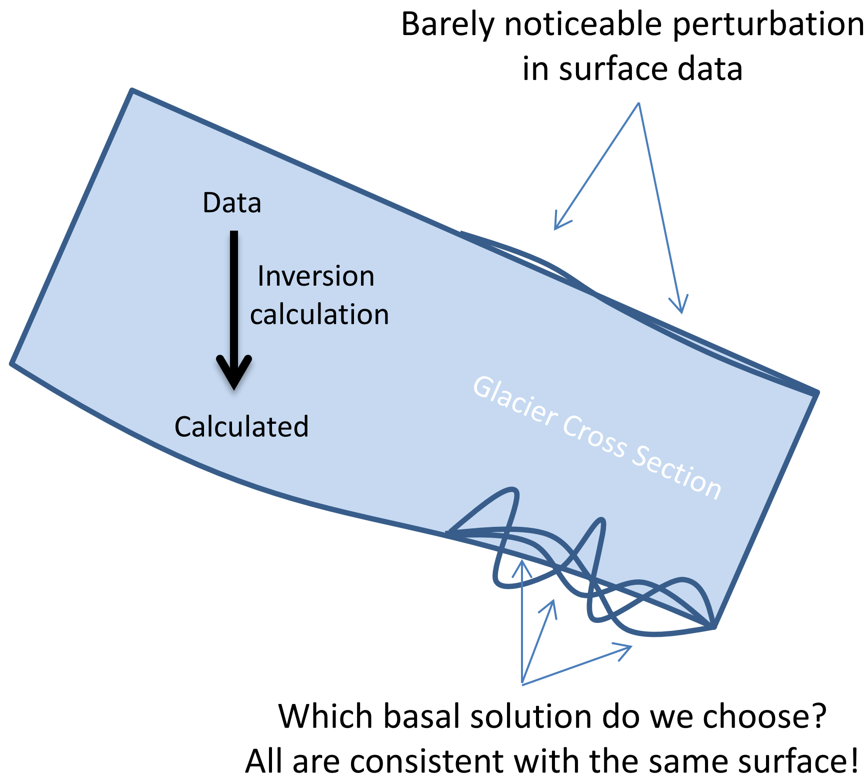
The Problem? Mathematical Inversion

- Basal conditions drive the surface conditions.
  - Slip at the bed determines velocity at surface.
  - Topography at bed causes bumps on the surface.
- So use observed "effect" to derive the "cause."
  - Backwards!
- Called an inversion.



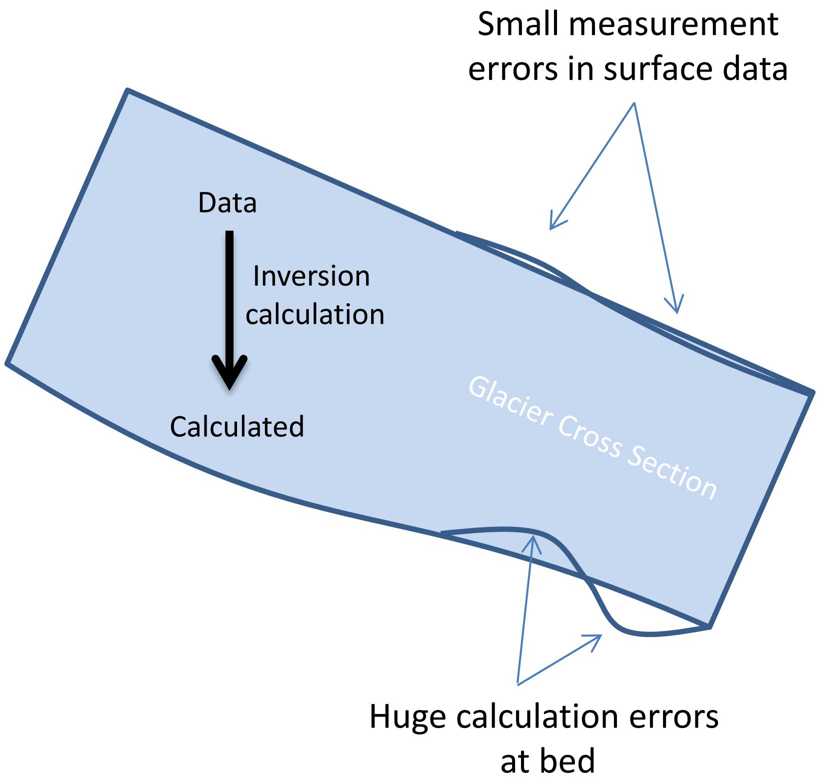
Cloud of Solutions

- Lots of bed conditions correspond to the same surface.
  - Different velocities, stresses, topographies…
  - Think of glacier as the viscous equivalent of a sponge. Wiggle the bottom of the sponge and will barely see it at the surface.
- So which one do we choose?
  - All fall within the measurement errors at surface.



Unstable Solution!

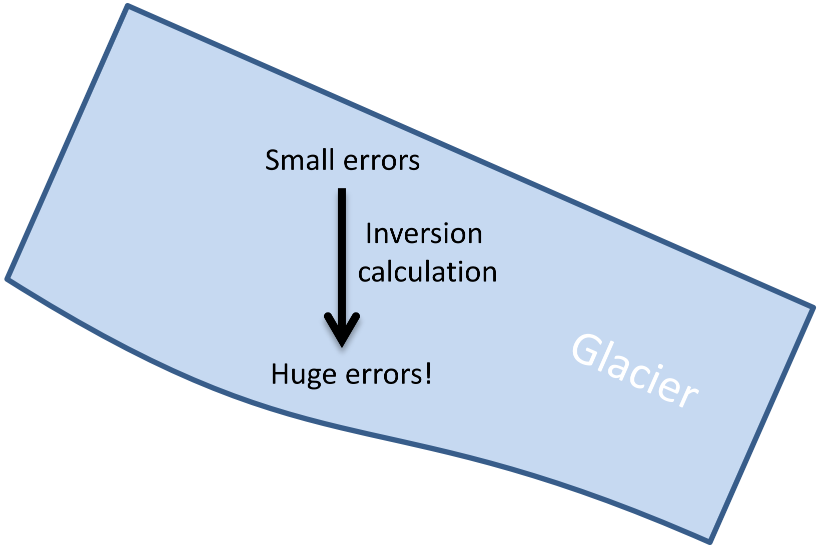
- Small errors in surface data translate to huge calculation errors at the bed.
  - Unstable.
- Note: We assume that nothing whatsoever is known about the bed.
  - This is the case for most GIC.
  - For ice sheets we often know basal topography and that can be a game changer.



Glacier Inversion is Ill-Posed

- Numerical inversions are often "ill-posed."
  - This does *not* mean we set up the problem incorrectly.
  - Most geophysical inversions are ill-posed.

- Ill-posed means the solution is unstable or not unique.
  - A cloud of possible solutions.
  - Mathematically impossible to pick the correct one.



Balise and Raymond (1985), Lliboutry (1987), MacAyeal (1993), Bahr et al (1994), Truffer (2004), Chandler et al (2006), and many more!

Ill-Posed Errors Grow Exponentially

Stress error at depth z

Stress error at surface

Evil exponential

$$\Delta\hat{\sigma}'_{ij}(k,z) = \Delta\hat{\sigma}'_{ij}(k,0)e^{0.6kz}$$

Error

Stress

Spatial frequency

Depth

Surface

The ^ means Fourier-transformed
to spatial frequency domain.

From Bahr et al (1994).

Worst at High Frequency And Deep Depths

$$\Delta\hat{\sigma}'_{ij}(k,z) = \Delta\hat{\sigma}'_{ij}(k,0)e^{0.6kz}$$

Exponential is huge at large k and z.

Exponential is small at small k.

Want $0.6kz \ll 1$.

Want Small Exponential

$$0.6\,k\,z \ll 1$$

Spatial frequency

$$\lambda \ll 0.6(2\pi)\,z$$

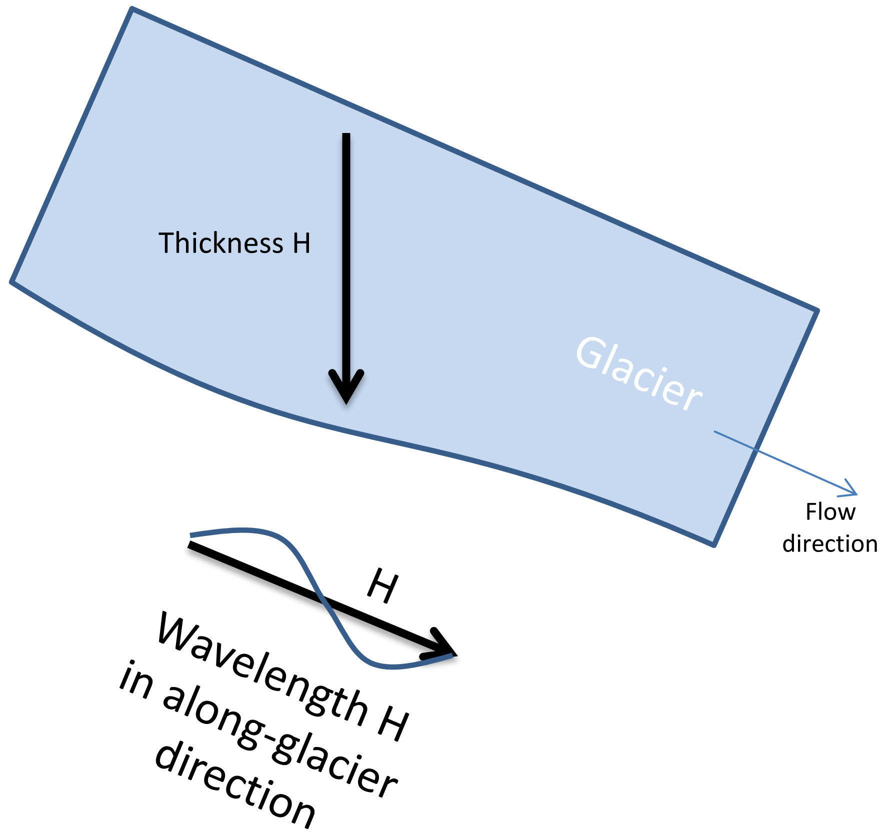
Equivalent spatial wavelength.

$$\lambda \ll 4\,z$$

Wavelength should be order of magnitude (10 times) smaller than 4 z.

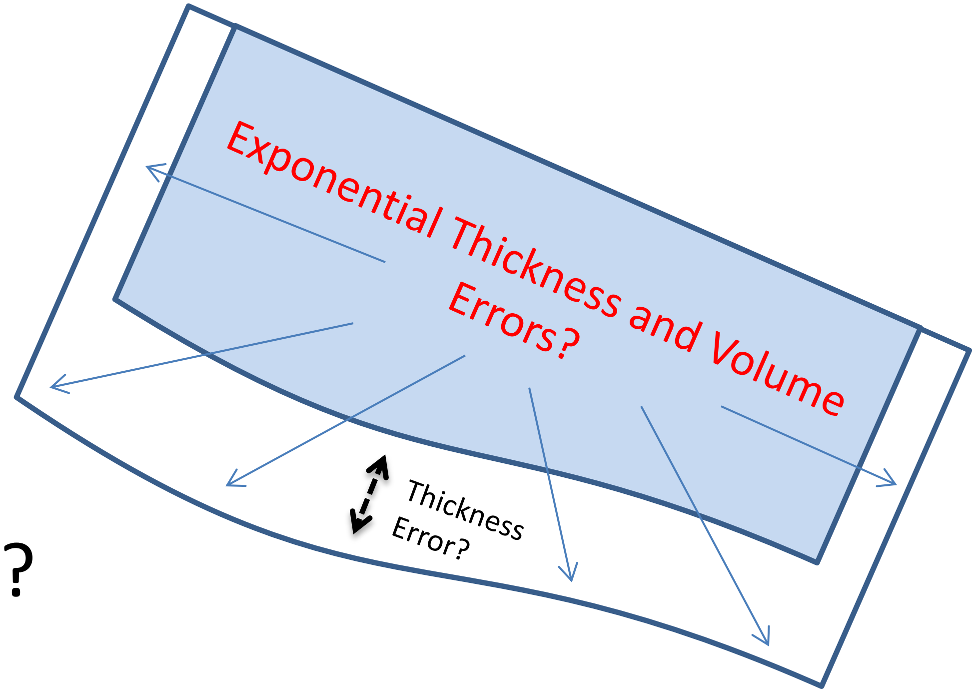
Practical Implication

- Want exponential to be negligible.
- High spatial frequencies cause big exponential.
- So remove all spatial wavelengths less than 40 H.
  - In along-glacier direction. Not depth.



But Wait! That's Just Stress?

- Stress errors grow exponentially.
  - From perturbation analysis of continuum mechanics.
  - Bahr et al (1994).
- What about velocity, thickness, volume, etc.?
  - Remember? We want volume errors for SLR!



Scaling Solution!

- All glaciological continuum parameters scale with all others.
  - From dimensional analysis.
  - Also from stretching transformations of continuum equations.
  - Bahr (1997), Bahr et al (1997), Bahr and Rundle (1995)
- Any one implies all others.
  - Observed are in blue.

$$V = cA^{\gamma}$$

$$V = c_v \sigma_{ij}'^{\gamma_v}$$

$$H = c_H \sigma_{ij}'^{\gamma_H}$$

$$A = c_A \sigma_{ij}'^{\gamma_A}$$

$$L = bA^{\beta}$$

$$\vdots$$

Lots of Little Devils in the Details

- But bottom line:
  Can transform stress solution to any other parameter.

  - Detail: Errors in surface stress are *always* there, either measured or calculated by model.

  - So measured/calculated errors in stress translate to errors in volume, thickness, etc.

$$\Delta\hat{\sigma}'_{ij} = \Delta\hat{\sigma}'_{ij}(k,0)e^{0.6kz}$$

$$\Delta\hat{V} \propto \Delta\hat{\sigma}'_{ij}(k,0)e^{0.6kz}$$

$$\Delta\hat{H} \propto \Delta\hat{\sigma}'_{ij}(k,0)e^{0.6kz}$$

Assorted scaling constants creep in here.

What's It Mean for Volume-Area Scaling?

- The wavelength that applies to scaling is $\lambda = 2L = 2A^{0.625}$.

- Substitute and get $\Delta\hat{V} = \Delta\hat{\sigma}'_{ij}(k,0)e^{0.064\,A^{-0.25}}$

- Effectively, no error growth!

Approaches zero for A > 1

What's It Mean For Numerical Inversions?

- Must filter volume solution at all wavelengths shorter than 40H.
- Suppose dx = along-glacier grid spacing of model.
  - $\lambda = 2dx$
  - Grid spacing must be 20H.
  - Can barely resolve small glaciers.
  - Will only have a few grid points in large glaciers.

Measured Area (km^2)	Expected Thickness (km)	Expected Length (km)	Grid Spacing dx (km)	Number of grid points dx in model
1	0.03	1	0.6	1
10	0.08	4	1.5	3
100	0.19	18	3.6	5
1000	0.45	75	8.5	9
10,000	1.08	316	20.3	16

Well, that's depressing.

What About Other Errors, Huh?

- Irrelevant compared to ill-posed errors.

  - Ill-posed errors are *exponential*.
  - Will swamp all other errors!

  - For example, $V = c\,A^{1.375}$, but *c* is poorly constrained. That means a *linear error* in *V* with linear error in *c*.
    - That's trivial compared to an exponential!

  - Unknown numerical model parameters. Ditto.

Upshot for Volume Calculations

- Can’t do much better than scaling.
  - Simple ☺
- Numerical models work just as well *if filtered*.
  - But sooo complicated ☹
- Lots of other sources of error.
  - Irrelevant if ill-posed errors are not controlled!
- Many good reasons to use models.
  - E.g., estimate englacial velocity and stress.
  - But volume is not one of them.