Glacial FOSLS

New FOSLS Formulation of Nonlinear Stokes Flow for Glaciers

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Outline:

1. Stokes for Glaciers
2. 2D Gravity Driven Glacier
3. Numerical Results

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Stokes for Glaciers

Viscosity Form

Continuity Equation:

$$\nabla \cdot \underline{u} = 0$$

Momentum Equation:

$$0 = \nabla \cdot \frac{1}{2}\mu\left(\nabla \underline{u} + (\nabla \underline{u})^T\right) - \nabla p + \rho \underline{g},$$

Viscosity

$$\mu = \frac{1}{2} A(T)^{-\frac{1}{3}} \dot{\varepsilon}_e^{-\frac{2}{3}},$$

$$\dot{\varepsilon}_e = ||\underline{\underline{\dot{\varepsilon}}}||_F,$$

$$\underline{\underline{\dot{\varepsilon}}} = \frac{1}{2}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right).$$

Rewrite as a First Order System

Definition

$$\underline{\underline{U}} = \nabla \underline{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\ \frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{21} & U_{31} \\ U_{12} & U_{22} & U_{32} \\ U_{13} & U_{23} & U_{33} \end{bmatrix}$$

H^1 Elliptic system

$$\nabla \cdot \underline{u} = 0 \qquad \text{(Continuity)}$$

$$\underline{\underline{U}} = \nabla \underline{u} \qquad \text{(Definition)}$$

$$\nabla \cdot \frac{1}{2}\mu \left(\underline{\underline{U}} + \underline{\underline{U}}^T\right) - \nabla p = -\rho \underline{g} \qquad \text{(Momentum)}$$

$$\nabla \times \underline{\underline{U}} = 0 \qquad \text{(Freebie)}$$

$$\text{Trace}(\underline{\underline{U}}) = 0 \qquad \text{(Enforced by setting } U_{11} = -U_{22}\text{)}$$

The biggest problem with this formulation comes in the definition for viscosity.

Viscosity

$$\mu = \frac{1}{2} A(T)^{-\frac{1}{3}} \dot{\varepsilon}_e^{-\frac{2}{3}}$$

$$\dot{\varepsilon}_e = ||\underline{\underline{\dot{\varepsilon}}}||_F$$

$$\underline{\underline{\dot{\varepsilon}}} = \frac{1}{2} \left(\nabla \underline{u} + (\nabla \underline{u})^T \right)$$

The viscosity is near infinite where the glacier experiences small deformations. This is usually overcome by using a small constant in the effective strain rate.

Definition

$$\underline{\underline{\hat{U}}} = \dot{\varepsilon}_e^{-\frac{2}{3}} \underline{\underline{U}}$$

$$\underline{\underline{\hat{\varepsilon}}} = \frac{1}{2}\left(\underline{\underline{\hat{U}}} + \underline{\underline{\hat{U}}}^T\right)$$

$$\hat{\varepsilon}_e = ||\underline{\underline{\hat{\varepsilon}}}||_F$$

$$\phi = \hat{\varepsilon}_e^2$$

Notice that

$$\phi = \hat{\varepsilon}_e^2 = ||\underline{\underline{\hat{\varepsilon}}}||_F^2 = ||\dot{\varepsilon}_e^{-\frac{2}{3}} \underline{\underline{\dot{\varepsilon}}}||_F^2 = \dot{\varepsilon}_e^{-\frac{4}{3}} ||\underline{\underline{\dot{\varepsilon}}}||_F^2 = \dot{\varepsilon}_e^{-\frac{4}{3}} \dot{\varepsilon}_e^2 = \dot{\varepsilon}_e^{\frac{2}{3}}$$

$$\phi = \frac{1}{4}\left((2\hat{U}_{11})^2 + 2(\hat{U}_{12} + \hat{U}_{21})^2 + (-2\hat{U}_{11})^2\right)$$

Stokes for Glaciers

Fluidity Form

Fluidity FOSLS Equations

$$
\begin{aligned}
\phi &= 2\hat{U}_{11}^2 + \frac{1}{2}\hat{U}_{12}^2 + \frac{1}{2}\hat{U}_{21}^2 + \hat{U}_{21}\hat{U}_{21} \\
\nabla \cdot \underline{u} &= 0 \\
\phi \underline{\underline{\hat{U}}} &= \nabla \underline{u} \\
\nabla \cdot \frac{1}{2} A(T)^{-\frac{1}{3}} \left(\underline{\underline{\hat{U}}} + \underline{\underline{\hat{U}}}^T \right) - \nabla p &= -\rho \underline{g} \\
\nabla \times \phi \underline{\underline{\hat{U}}} &= 0 \\
U_{11} &= -U_{22}
\end{aligned}
$$

Stokes for Glaciers

Scaling

When ϕ is small, the Div and Curl equations are not of the same scale.

Scaled Curl Equation

$$\frac{1}{\phi + c} \nabla \times \phi \underline{\underline{\hat{U}}} = 0$$

This is equivalent to:

Log Form (via: Product Rule)

$$\nabla \times \underline{\underline{\hat{U}}} - (\nabla^{\perp} \log(\phi + c)) \cdot \underline{\underline{\hat{U}}} = 0$$

Log Form has unscaled Curl equation with lower order terms.

Outline:

2D Gravity Driven Glacier

Parameters

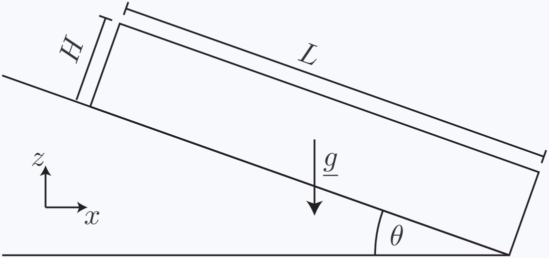
Parameters

- Bed Slope (θ) = 0.05
- $A(T) = 4 \times 10^{-24}$ Pa^{-3}s^{-1}
- $|g| = 9.81$ m/s^2
- $\rho = 900$ kg/m^3
- $H = 1000$ m
- $L = 10000$ m
- $n = 3$
- Assume pressure is zero on the surface

2D Gravity Driven Glacier

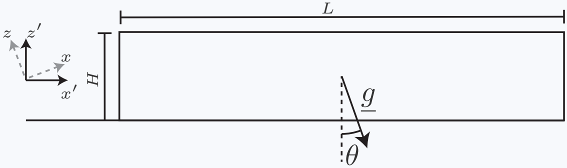
Cross Section

Domain



where θ is the bed slope $\underline{g} = |g|[0, -1]^T$

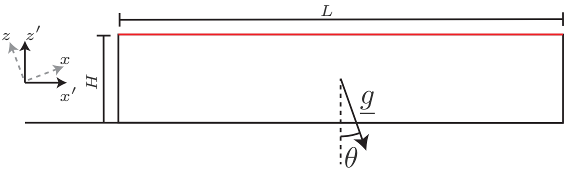
Computational Domain



now $\underline{g} = |g|[\sin(\theta), -\cos(\theta)]^T$

2D Gravity Driven Glacier

Boundary Conditions: Top

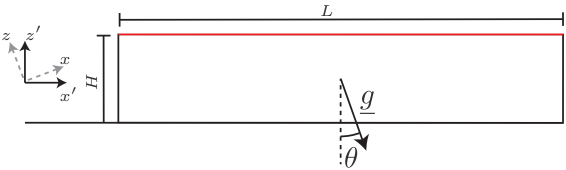


For the Top boundary we want to impose a stress free condition

$$\underline{\underline{\sigma}} \cdot \underline{n} = 0$$

$$\underline{\underline{\sigma}} = \underline{\underline{\hat{U}}} + \underline{\underline{\hat{U}}}^T - pI$$

$$\begin{bmatrix} 2\hat{U}_{11} - p & \hat{U}_{12} + \hat{U}_{21} \\ \hat{U}_{21} + \hat{U}_{12} & 2\hat{U}_{22} - p \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \hat{U}_{12} + \hat{U}_{21} \\ 2\hat{U}_{22} - p \end{bmatrix} = 0$$



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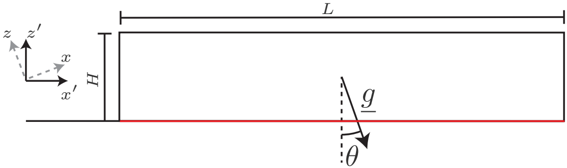


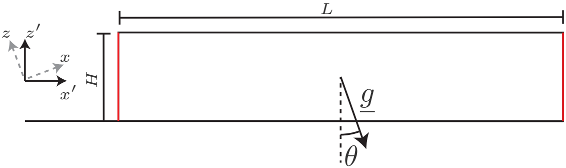
Assume the glacier is frozen to the bed (No Slip)

$$\underline{u} = 0$$

This also gives us

$$U_{11} = 0 \qquad U_{21} = 0$$
$$\hat{U}_{11} = 0 \qquad \hat{U}_{21} = 0$$

Finally, assume the periodic side boundaries.



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2D Gravity Driven Glacier

Exact Solution

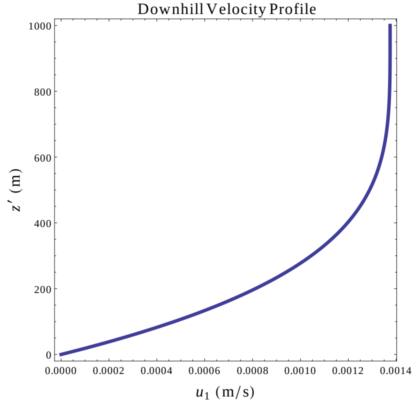
Notice that

$$\underline{u} = [u_1, u_2]^T = [f(z'), 0]^T$$

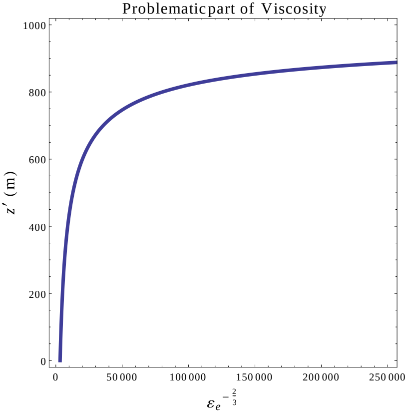
Using this and the other assumptions, we can backtrack to find the exact solution:

Solution

$$\begin{aligned}
u_1 &= A(\rho|g|\sin(\theta))^3(H^4 - (H-z)^4), \\
U_{12} &= 4A(\rho|g|\sin(\theta))^3(H-z)^3, \\
p &= \rho|g|\cos(\theta)(H-z), \\
\phi &= 2A^{\frac{2}{3}}(\rho|g|\sin(\theta))^2(H-z)^2, \\
\hat{U}_{12} &= 2A^{\frac{1}{3}}(\rho|g|\sin(\theta))(H-z), \\
u_2 &= \hat{U}_{11} = \hat{U}_{21} = \hat{U}_{22} = 0.
\end{aligned}$$



Downhill Velocity Profile
1000
800
600
400
200
0
z′ (m)
0.0000
0.0002
0.0004
0.0006
0.0008
0.0010
0.0012
0.0014
u₁ (m/s)

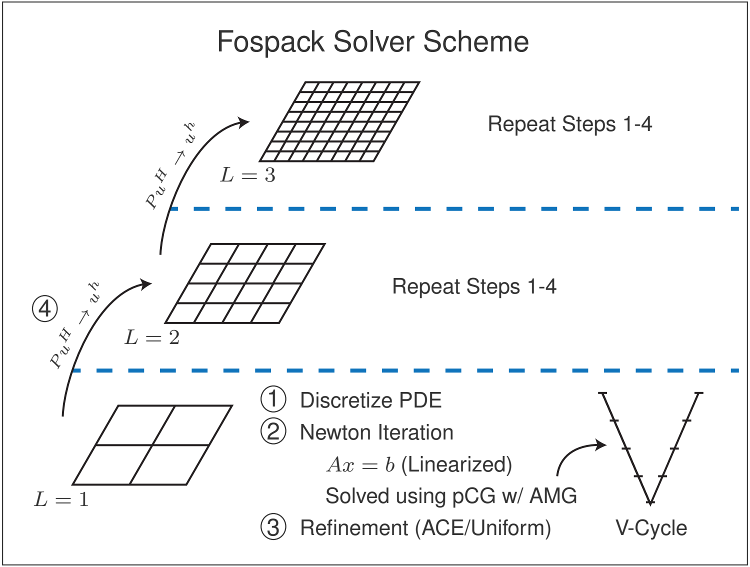


Problematic part of Viscosity
1000
800
600
400
200
0
z′ (m)
0
50 000
100 000
150 000
200 000
250 000

Outline:

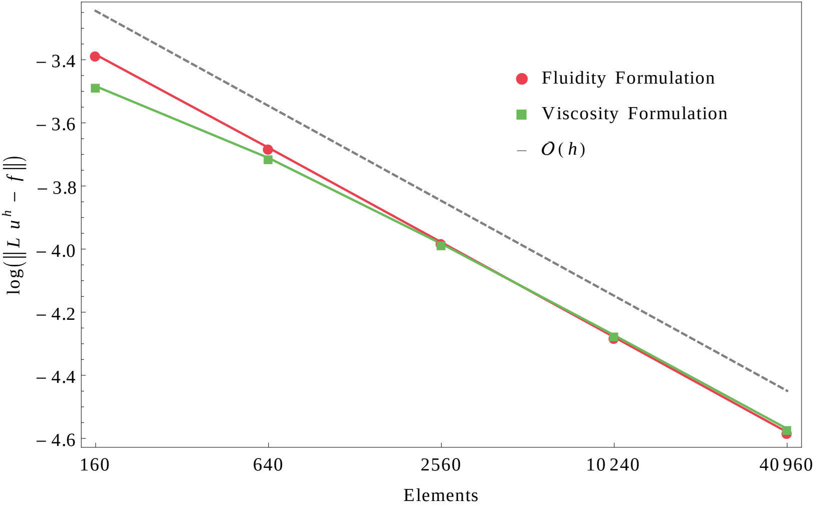
Numerical Results

Solver (Fospack)



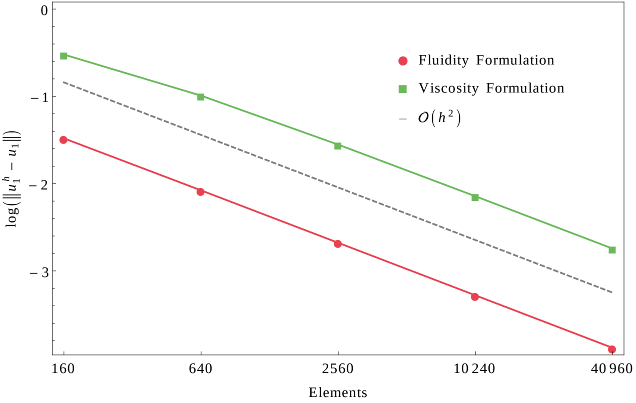
Numerical Results

Functional Reduction



Numerical Results

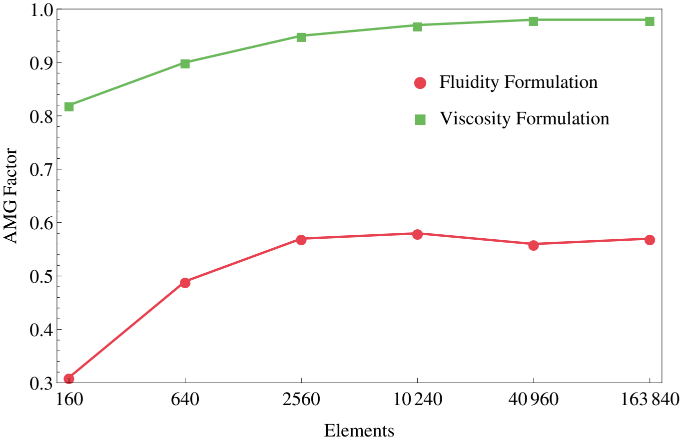
L^2 Reduction



Final Error:
Viscosity Formulation: 1.8×10^{-3} (14 times larger)
Fluidity Formulation: 1.3×10^{-4}

Numerical Results

AMG Factor: Comparison



$$0.98^q = 0.58 \qquad q \approx 27$$

Numerical Results

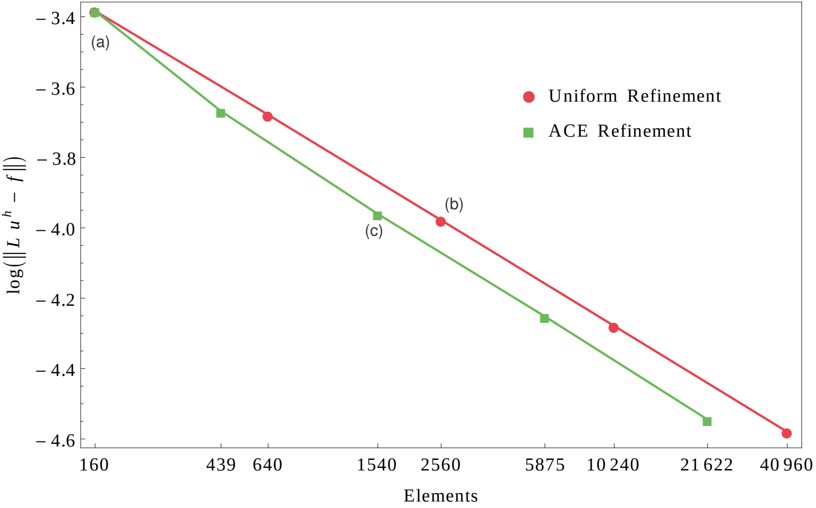
Work Table

Summary of the fluidity formulation's numerical preformance. L is the level of refinment. N is the number of Newton steps. Complexity lists the cycle complexity for each Newton step. WU is the total number of work units for that level. Functional refers to the nonlinear functional norm.

Level	E	Nonzeros	N	Complexity	V-Cycles	WU	Functional
1	160	52000	2	3.59, 3.79	8, 7	0.121	4.13×10^{-4}
2	640	196000	1	4.10	5	0.170	2.10×10^{-4}
3	2560	760480	1	4.44	4	0.570	1.05×10^{-4}
4	10240	2995360	1	4.60	3	1.745	5.25×10^{-5}
5	40960	11888800	1	4.72	3	7.108	2.63×10^{-5}
6	163840	47370400	1	4.80	3	28.800	1.32×10^{-5}
					Total	38.514	

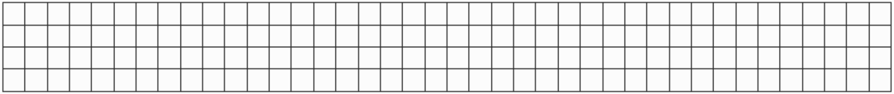
Numerical Results

Functional: Uniform Vs. ACE

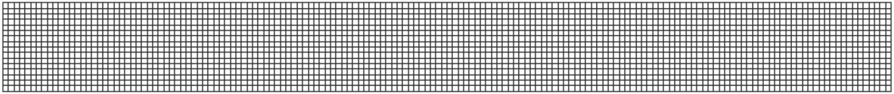


Numerical Results

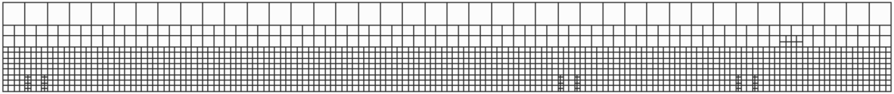
ACE Grids



(a)



(b)



(c)

Conclusions

1. Glaciers are modeled By Stokes with nonlinear viscosity.
2. Viscosity become nearly infinite when the glacier experiences small deformations.
3. Nonlinear FOSLS formulation captures the physical behavior.
4. The fluidity formulation yields better numerical performance.

Future Work

- Benchmark Problems (ISMIP)
- Inclusion of Energy Model
- Time Dependent Domain

Questions?

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Thanks to the Grandview Gang and Glaciers Group
Special Thanks to Tom, Steve, and Hari