Free boundaries and conservation equations in ice sheet models

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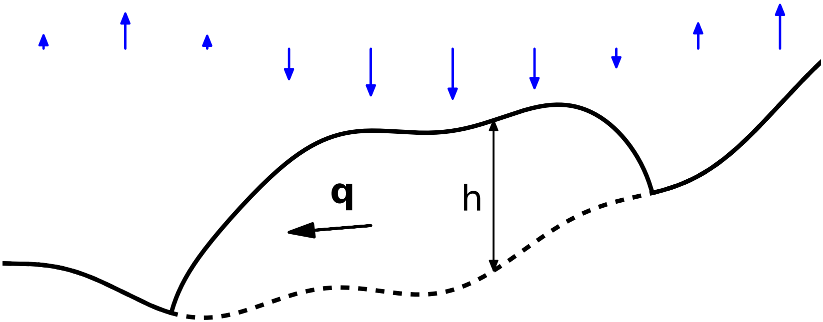
Outline

1. The problem I'm worried about:
   - Time-stepping free-boundary fluid layer models.
2. Practical consequences:
   - Limitations to discrete conservation.
   - Need for weak numerical free boundary solutions.

A fluid layer in a climate

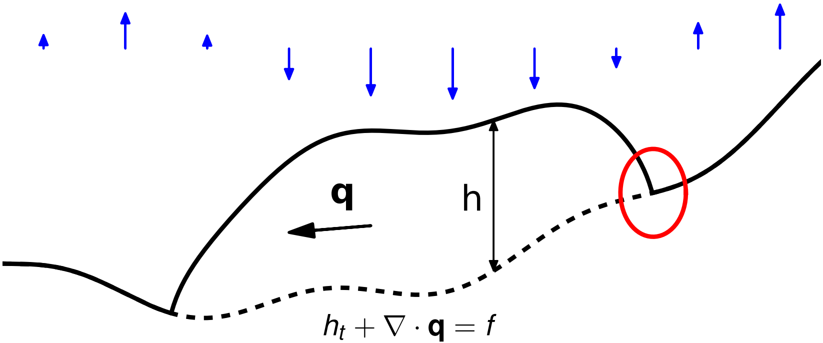


- mass conservation PDE for a layer:

$$h_t + \nabla \cdot \mathbf{q} = f$$

  - h is a thickness: $h \geq 0$
  - mass conservation PDE applies *only where* $h > 0$
  - $\mathbf{q}$ is flow (vertically-integrated)
  - source f is “climate”; $f > 0$ shown downward

A fluid layer in a climate: *the troubles*



- $h = 0$ *and what else* at free boundary?
  - shape at free boundary depends on both $\mathbf{q}$ and f
- $f < 0$ not "detected" by model where $h = 0$
  - how to do mass conservation accounting?
- $f \approx 0$ threshold behavior
  - $h > 0$ as soon as $f < 0$ switches to $f > 0$

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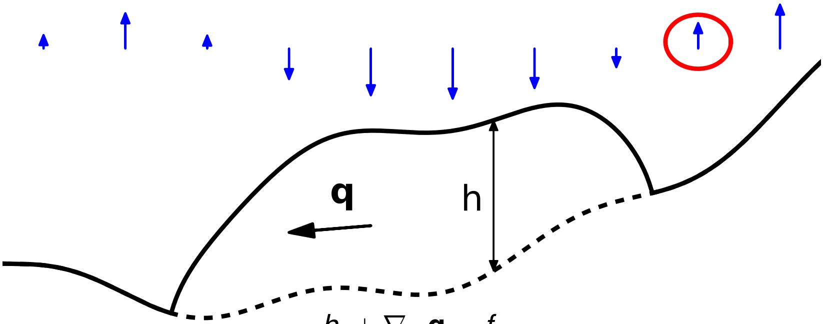

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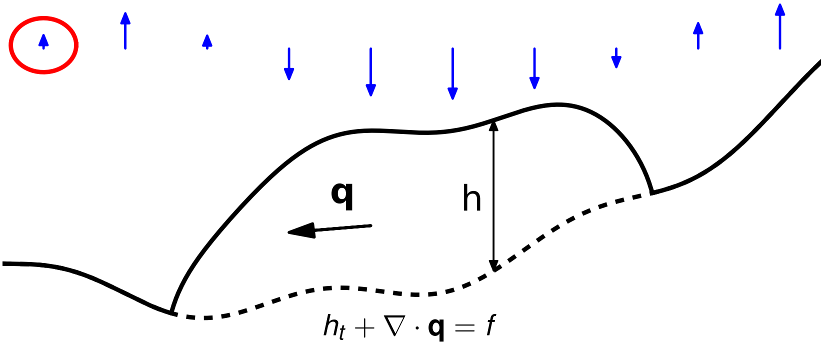

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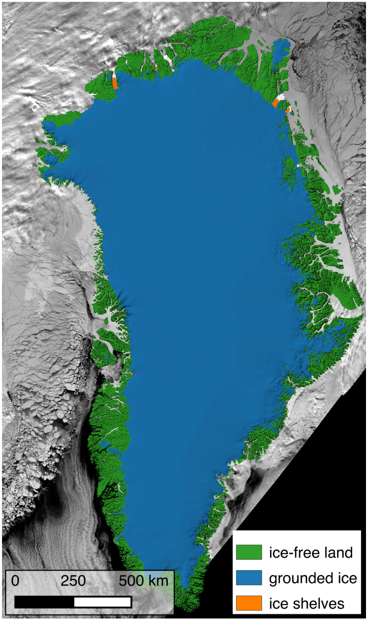
A concern driven by practical modeling

- the icy region is nearly-fractal and disconnected
- currently in PISM*:
  - explicit time-stepping
  - free boundary by truncation
- want for PISM:
  - implicit time steps
  - better conservation accounting to user

*= Parallel Ice Sheet Model, `pism-docs.org`



Examples

glaciers

ice shelves & sea ice

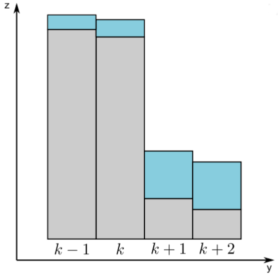
tidewater marsh

tsunami inundation

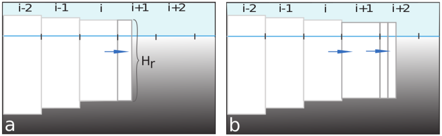
and subglacial hydrology, supraglacial runoff, surface hydrology, . . .

Anyone numerically-solved these problems before?

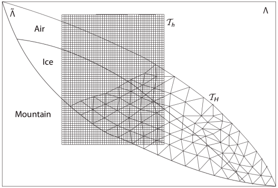
- yes, of course!
  - generic result: *ad hoc* schemes near the free boundary



glacier ice
on steep terrain
(Jarosch, Schoof, Anslow, 2013)



volume-of-fluid method at ice shelf fronts
(Albrecht et al, 2011)



volume-of-fluid method at glacier surface
(Jouvet et al 2008)

New goals

- I don't mind "`if ...then ...`" in my code, *but* I want to know what mathematical problem is behind it
  - maintaining code with those *ad hoc* schemes scares the #!*& out of me
- my goals:
  - redefine the problem so free boundary is part of solution
  - tell the model user what is going on at the free boundary
  - find numerical schemes which automate the details

Numerical models *must* discretize time

$$h_t + \nabla \cdot \mathbf{q} = f \qquad \to \qquad \frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

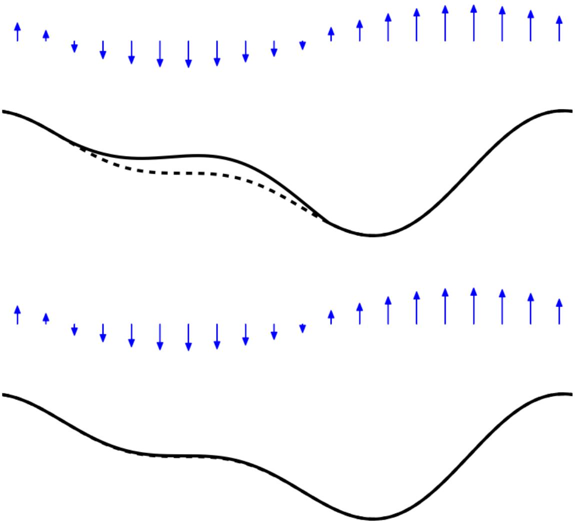
- semi-discretize in time: $H_n(x) \approx h(t_n, x)$
- the new equation is a "single time-step problem"
  - a PDE in space where $H_n > 0$
  - called the "strong form"
- details of flux $\mathbf{Q}_n$ and source F_n come from time-stepping scheme

1D time-stepping examples

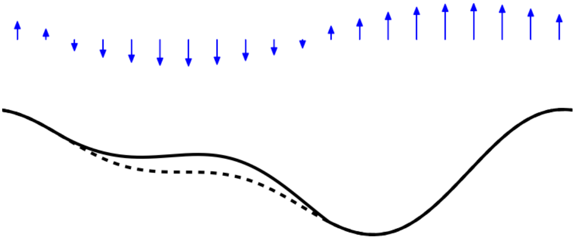
same:

- equation $\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = f$
- climate f
- bed shape
- constrained-Newton scheme

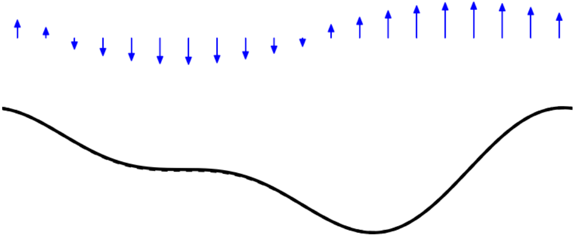
how different are the fluxes $\mathbf{Q}_n$?



1D time-stepping examples



$\mathbf{Q}_n = v_0 H_n$
hyperbolic
(constant velocity)

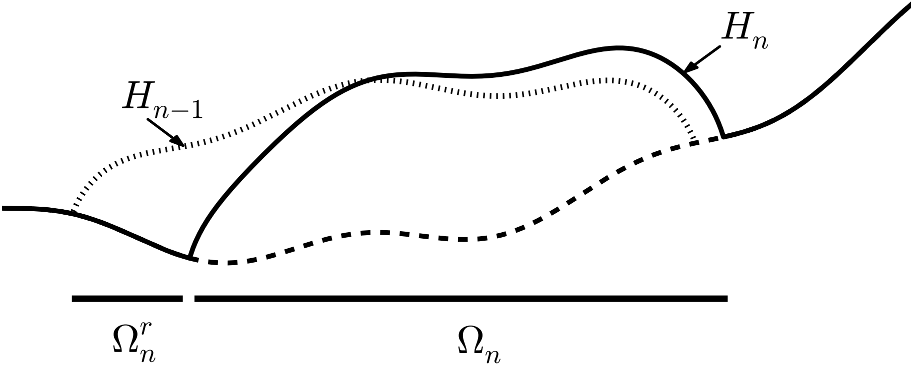


$\mathbf{Q}_n = -\Gamma |H_n|^{n+2} \cdot |\nabla h_n|^{n-1} \nabla h_n$
highly-nonlinear diffusion

Subsets for time-stepping and conservation

- suppose H_n solves the single time-step problem
- define

$$\Omega_n = \{H_n(x) > 0\}$$
$$\Omega_n^r = \{H_n(x) = 0 \text{ and } H_{n-1}(x) > 0\} \qquad \leftarrow \text{retreat set}$$



Reporting discrete conservation

- define:

$$M_n = \int_\Omega H_n(x)\,dx \qquad \textit{mass at time } t_n$$

- then

$$\boxed{\Delta t \left(-\nabla \cdot \mathbf{Q}_n + F_n\right)}$$

$$\begin{aligned} M_n - M_{n-1} &= \int_{\Omega_n} \boxed{H_n - H_{n-1}}\,dx + \int_{\Omega_n^r} 0 - H_{n-1}\,dx \\ &= \Delta t \left(0 + \int_{\Omega_n} F_n\,dx\right) - \int_{\Omega_n^r} H_{n-1}\,dx \end{aligned}$$

- new term:

$$R_n = \int_{\Omega_n^r} H_{n-1}\,dx \qquad \textit{retreat loss during step } n$$

Reporting discrete conservation: *limitation*

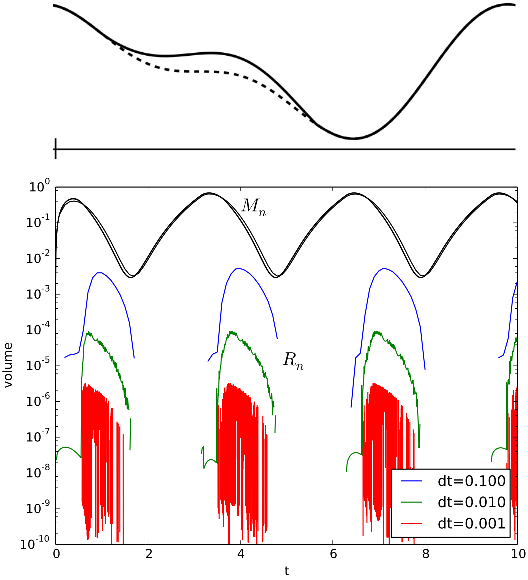
- the retreat loss R_n is not balanced by the climate
  - R_n is *caused* by the climate, but we don't know a *computable integral* to balance it
- we must track three time series:
  - mass at time t_n: $M_n = \int_\Omega H_n(x)\, dx$
  - climate (e.g. surface mass bal.) over current fluid-covered region:

$$C_n = \Delta t \int_{\Omega_n} F_n\, dx \approx \int_{t_{n-1}}^{t_n} \int_{\Omega_n} f(t,x)\, dx\, dt$$

  - retreat loss during time step: $R_n = \int_{\Omega_n^r} H_{n-1}\, dx$
- now it balances:

$$M_n = M_{n-1} + C_n - R_n$$

Reporting discrete conservation: $R_n \to 0$ as $\Delta t \to 0$



Weak form incorporates constraint

- define:

$$\mathcal{K} = \left\{ v \in W^{1,p}(\Omega) \,\middle|\, v \geq 0 \right\} = \text{admissible thicknesses}$$

- we say $H_n \in \mathcal{K}$ solves the weak single time-step problem if

$$\int_\Omega H_n(v - H_n) - \Delta t\, \mathbf{Q}_n \cdot \nabla(v - H_n) \geq \int_\Omega (H_{n-1} + \Delta t\, F_n)\,(v - H_n)$$

  for all $v \in \mathcal{K}$

  - derive this *variational inequality* from:
    - the strong form *and*
    - integration-by-parts *and*
    - arguments about $H_n = 0$ areas

Weak solves strong, *and* it gives more info

- assume $\mathbf{Q}_n = 0$ when $H_n = 0$
  - this means $\mathbf{Q}_n$ describes a *layer*
- assume $H_n \in \mathcal{K}$ solves weak single time-step problem
- then
  1. PDE applies on the set where $H_n > 0$:

$$\frac{H_n - H_{n-1}}{\Delta t} + \nabla \cdot \mathbf{Q}_n = F_n$$

  2. information on the set where $H_n = 0$:

$$H_{n-1} + \Delta t \, F_n \leq 0$$

     - this means "mass balance was negative enough during time step to remove old thickness"

Numerical solution of the weak problem

the weak single time-step problem:

- is nonlinear because of constraint (even for $\mathbf{Q}_n$ linear in H_n)
- can be solved by a Newton method modified for constraint
    - reduced set method
    - semismooth method
- scalable implementations are in PETSc 3.5
    - see "SNESVI" object

Summary

- layer flow model has conservation eqn. $h_t + \nabla \cdot \mathbf{q} = f$
  - long time steps wanted, but this is a free-boundary problem ...
- claim: exact discrete conservation requires tracking *retreat loss*
  - in addition to computable integrals of climate
  - it only disappears in $\Delta t \to 0$ limit
- suggestions:
  - *include* constraint on thickness: $h \geq 0$
  - pose single time-step problem *weakly* as variational inequality
  - solve it numerically by constrained-Newton method
- these are agnostic claims/suggestions, with respect to:
  - form of the flux $\mathbf{q}$
  - spatial discretization paradigm (i.e. finite diff./volume/element)