A Fluidity Based First-Order System Least-Squares Method for Ice Sheets

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Outline:

1. FOSLS Formulations
2. Test Problems
3. Future Plans

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1. FOSLS Formulations
   - Viscosity Formulation
   - Fluidity Formulation
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Stokes for Glaciers

Quick Reminder

Continuity Equation:

$$\nabla \cdot \underline{u} = 0$$

Momentum Equation:

$$0 = \nabla \cdot \mu \left(\nabla \underline{u} + (\nabla \underline{u})^T \right) - \nabla p + \rho \underline{g},$$

Viscosity

$$\mu = \frac{1}{2} \left(\frac{A}{2} \right)^{-\frac{1}{3}} ||\underline{\underline{\dot{\varepsilon}}}||_F^{-\frac{2}{3}} ,$$

$$\underline{\underline{\dot{\varepsilon}}} = \frac{1}{2} \left(\nabla \underline{u} + (\nabla \underline{u})^T \right) .$$

Rewrite as a First Order System

Definition

$$\underline{\underline{U}} = \nabla \underline{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\ \frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{21} & U_{31} \\ U_{12} & U_{22} & U_{32} \\ U_{13} & U_{23} & U_{33} \end{bmatrix}$$

First Order System:

$$\nabla \cdot \underline{u} = 0 \qquad \text{(Continuity)}$$

$$\underline{\underline{U}} = \nabla \underline{u} \qquad \text{(Definition)}$$

$$\nabla \cdot \frac{1}{2}\mu \left(\underline{\underline{U}} + \underline{\underline{U}}^T\right) - \nabla p = -\rho \underline{g} \qquad \text{(Momentum)}$$

Rewrite as a First Order System

Definition

$$\underline{\underline{U}} = \nabla \underline{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\ \frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{21} & U_{31} \\ U_{12} & U_{22} & U_{32} \\ U_{13} & U_{23} & U_{33} \end{bmatrix}$$

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$$\nabla \times \underline{\underline{U}} = 0 \qquad \text{(Curl of Definition)}$$

Rewrite as a First Order System

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$$\underline{\underline{U}} = \nabla \underline{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\ \frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{21} & U_{31} \\ U_{12} & U_{22} & U_{32} \\ U_{13} & U_{23} & U_{33} \end{bmatrix}$$

First Order System:

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$$\nabla \cdot \frac{1}{2}\mu \left(\underline{\underline{U}} + \underline{\underline{U}}^T\right) - \nabla p = -\rho \underline{g} \qquad \text{(Momentum)}$$

$$\nabla \times \underline{\underline{U}} = 0 \qquad \text{(Curl of Definition)}$$

$$\text{Trace}(\underline{\underline{U}}) = 0$$

Rewrite as a First Order System

Definition

$$\underline{\underline{U}} = \nabla \underline{u} = \begin{bmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_2}{\partial x} & \frac{\partial u_3}{\partial x} \\ \frac{\partial u_1}{\partial y} & \frac{\partial u_2}{\partial y} & \frac{\partial u_3}{\partial y} \\ \frac{\partial u_1}{\partial z} & \frac{\partial u_2}{\partial z} & \frac{\partial u_3}{\partial z} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{21} & U_{31} \\ U_{12} & U_{22} & U_{32} \\ U_{13} & U_{23} & U_{33} \end{bmatrix}$$

Viscosity Formulation:

$$\begin{aligned} \nabla \cdot \underline{u} &= 0 && \text{(Continuity)} \\ \underline{\underline{U}} &= \nabla \underline{u} && \text{(Definition)} \\ \nabla \cdot \frac{1}{2}\mu \left(\underline{\underline{U}} + \underline{\underline{U}}^T\right) - \nabla p &= -\rho \underline{g} && \text{(Momentum)} \\ \nabla \times \underline{\underline{U}} &= 0 && \text{(Curl of Definition)} \\ \text{Trace}(\underline{\underline{U}}) &= 0 && \text{(Enforced by setting } U_{11} = -U_{22}\text{)} \end{aligned}$$

Outline:

1. FOSLS Formulations
   - Viscosity Formulation
   - Fluidity Formulation
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Fluidity Formulation

Problem

A problem with this formulation comes in the definition for viscosity.

Viscosity

$$\mu = \frac{1}{2} A^{-\frac{1}{3}} \dot{\varepsilon}_e^{-\frac{2}{3}}$$

$$\dot{\varepsilon}_e = \frac{1}{\sqrt{2}} ||\underline{\underline{\dot{\varepsilon}}}||_F$$

$$\underline{\underline{\dot{\varepsilon}}} = \frac{1}{2} \left(\nabla \underline{u} + (\nabla \underline{u})^T \right)$$

The viscosity is near infinite where the glacier experiences small deformations. This is usually overcome by using a small constant in the effective strain rate.

Fluidity Formulation

Move Constants

Viscosity

$$\mu = c_A \left|\left|\underline{\underline{\dot{\varepsilon}}}\right|\right|_F^{-\frac{2}{3}},$$

$$c_A = \frac{1}{2}\left(\frac{A}{2}\right)^{-\frac{1}{3}},$$

Momentum Equation:

$$0 = \nabla \cdot \hat{\mu}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right) - \nabla \hat{p} + \hat{\rho}\underline{g},$$

$$\hat{\mu} = \left|\left|\underline{\underline{\dot{\varepsilon}}}\right|\right|_F^{-\frac{2}{3}}, \qquad \hat{p} = \frac{p}{c_A}, \qquad \hat{\rho} = \frac{\rho}{c_A}.$$

Fluidity Formulation

Different FOS-ification

Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

$$\underline{\underline{\sigma'}} = 2\hat{\mu}\underline{\underline{\dot{\varepsilon}}}$$

$$= \hat{\mu}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right)$$

$$\begin{bmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{12} & \sigma'_{22} \end{bmatrix} = \begin{bmatrix} 2\hat{\mu}\partial_x u_1 & \hat{\mu}(\partial_y u_1 + \partial_x u_2) \\ \hat{\mu}(\partial_y u_1 + \partial_x u_2) & 2\hat{\mu}\partial_y u_2 \end{bmatrix}$$

Fluidity Formulation

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Fluidity Formulation

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Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

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\underline{\underline{\sigma'}} &= 2\hat{\mu}\underline{\underline{\dot{\varepsilon}}} \\
&= \hat{\mu}\left(\nabla \underline{u} + (\nabla \underline{u})^T\right)
\end{aligned}
$$

$$
\begin{bmatrix} \sigma'_{11} & \sigma'_{12} \\ \sigma'_{12} & -\sigma'_{11} \end{bmatrix} = \begin{bmatrix} 2\hat{\mu}\partial_x u_1 & \hat{\mu}(\partial_y u_1 + \partial_x u_2) \\ \hat{\mu}(\partial_y u_1 + \partial_x u_2) & 2\hat{\mu}\partial_y u_2 \end{bmatrix}
$$

Using the Continuity Equation, $2\partial_x u_1$ can be split into $\partial_x u_1 - \partial_y u_2$

Fluidity Formulation

Different FOS-ification

Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

$$\begin{aligned}\underline{\underline{\sigma'}} &= 2\hat{\mu}\underline{\underline{\dot{\varepsilon}}} \\ &= \hat{\mu}\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)\end{aligned}$$

$$\begin{bmatrix}\sigma'_{11} & \sigma'_{12} \\ \sigma'_{12} & -\sigma'_{11}\end{bmatrix} = \begin{bmatrix}\hat{\mu}(\partial_x u_1 - \partial_y u_2) & \hat{\mu}(\partial_y u_1 + \partial_x u_2) \\ \hat{\mu}(\partial_y u_1 + \partial_x u_2) & \hat{\mu}(\partial_y u_2 - \partial_x u_1)\end{bmatrix}$$

Fluidity Formulation

Different FOS-ification

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$$
\begin{aligned}
\underline{\underline{\sigma'}} &= 2\hat{\mu}\underline{\underline{\dot{\varepsilon}}} \\
&= \hat{\mu}\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)
\end{aligned}
$$

$$
\begin{bmatrix} \hat{\mu}^{-1}\sigma'_{11} & \hat{\mu}^{-1}\sigma'_{12} \\ \hat{\mu}^{-1}\sigma'_{12} & -\hat{\mu}^{-1}\sigma'_{11} \end{bmatrix} = \begin{bmatrix} \partial_x u_1 - \partial_y u_2 & \partial_y u_1 + \partial_x u_2 \\ \partial_y u_1 + \partial_x u_2 & \partial_y u_2 - \partial_x u_1 \end{bmatrix}
$$

define the fluidity as $\phi = \hat{\mu}^{-1}$

Fluidity Formulation

Different FOS-ification

Another way to turn Stokes for glaciers into a first order system is to start with the deviatoric stress.

$$\begin{aligned}\underline{\underline{\sigma'}} &= 2\hat{\mu}\underline{\underline{\dot{\varepsilon}}} \\ &= \hat{\mu}\left(\nabla\underline{u} + (\nabla\underline{u})^T\right)\end{aligned}$$

$$\begin{bmatrix}\phi\sigma'_{11} & \phi\sigma'_{12} \\ \phi\sigma'_{12} & -\phi\sigma'_{11}\end{bmatrix} = \begin{bmatrix}\partial_x u_1 - \partial_y u_2 & \partial_y u_1 + \partial_x u_2 \\ \partial_y u_1 + \partial_x u_2 & \partial_y u_2 - \partial_x u_1\end{bmatrix}$$

Now we have the two equations:

$$\phi\sigma'_{11} - (\partial_x u_1 - \partial_y u_2) = 0$$

$$\phi\sigma'_{12} - (\partial_y u_1 + \partial_x u_2) = 0$$

Fluidity Formulation

Definition of Fluidity

Now we just need to figure out what ϕ is in terms of σ'_{11} and σ'_{12}.

$$
\begin{aligned}
\hat{\mu} &= ||\underline{\underline{\dot{\varepsilon}}}||_F^{-\frac{2}{3}} \\
&= ||\underline{\underline{\dot{\varepsilon}}}||_F^{-\frac{4}{3}} ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= \hat{\mu}^2 ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\hat{\mu}\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\underline{\underline{\sigma'}}/2||_F^2 \\
&= \frac{1}{2}({\sigma'_{11}}^2 + {\sigma'_{12}}^2)
\end{aligned}
$$

Fluidity Formulation

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$$\begin{aligned}\hat{\mu}^{-1} &= ||\underline{\underline{\dot{\varepsilon}}}||_F^{\frac{2}{3}} \\ &= ||\underline{\underline{\dot{\varepsilon}}}||_F^{-\frac{4}{3}} ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\ &= \hat{\mu}^2 ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\ &= ||\hat{\mu}\underline{\underline{\dot{\varepsilon}}}||_F^2 \\ &= ||\underline{\underline{\sigma'}}/2||_F^2 \\ &= \frac{1}{2}({\sigma'_{11}}^2 + {\sigma'_{12}}^2)\end{aligned}$$

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\hat{\mu}^{-1} &= ||\underline{\underline{\dot{\varepsilon}}}||_F^{\frac{2}{3}} \\
&= ||\underline{\underline{\dot{\varepsilon}}}||_F^{-\frac{4}{3}} ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
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&= ||\hat{\mu}\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\underline{\underline{\sigma'}}/2||_F^2 \\
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$$

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&= ||\underline{\underline{\dot{\varepsilon}}}||_F^{-\frac{4}{3}} ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= \hat{\mu}^2 ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\hat{\mu}\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\underline{\underline{\sigma'}}/2||_F^2 \\
&= \frac{1}{2}({\sigma'_{11}}^2 + {\sigma'_{12}}^2)
\end{aligned}
$$

Fluidity Formulation

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&= \hat{\mu}^2 ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\hat{\mu}\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\underline{\underline{\sigma'}}/2||_F^2 \\
&= \frac{1}{2}({\sigma'_{11}}^2 + {\sigma'_{12}}^2)
\end{aligned}
$$

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$$
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&= \hat{\mu}^2 ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\hat{\mu}\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\underline{\underline{\sigma'}}/2||_F^2 \\
&= \frac{1}{2}({\sigma'_{11}}^2 + {\sigma'_{12}}^2)
\end{aligned}
$$

Fluidity Formulation

Definition of Fluidity

Now we just need to figure out what ϕ is in terms of σ'_{11} and σ'_{12}.

$$
\begin{aligned}
\hat{\mu}^{-1} &= ||\underline{\underline{\dot{\varepsilon}}}||_F^{\frac{2}{3}} \\
&= ||\underline{\underline{\dot{\varepsilon}}}||_F^{-\frac{4}{3}} ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= \hat{\mu}^2 ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\hat{\mu}\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\underline{\underline{\sigma'}}/2||_F^2 \\
&= \frac{1}{2}({\sigma'_{11}}^2 + {\sigma'_{12}}^2)
\end{aligned}
$$

Now we just need to figure out what ϕ is in terms of σ'_{11} and σ'_{12}.

$$\begin{aligned}
\hat{\mu}^{-1} &= ||\underline{\underline{\dot{\varepsilon}}}||_F^{\frac{2}{3}} \\
&= ||\underline{\underline{\dot{\varepsilon}}}||_F^{-\frac{4}{3}} ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= \hat{\mu}^2 ||\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\hat{\mu}\underline{\underline{\dot{\varepsilon}}}||_F^2 \\
&= ||\underline{\underline{\sigma'}}/2||_F^2 \\
\phi &= \frac{1}{2}({\sigma'_{11}}^2 + {\sigma'_{12}}^2)
\end{aligned}$$

Fluidity Formulation

Stress-Gradient-Fluidity

Stress-Gradient-Fluidity Formulation

$$\phi\sigma'_{11} - (\partial_x u_1 - \partial_y u_2) = 0,$$
$$\phi\sigma'_{12} - (\partial_y u_1 + \partial_x u_2) = 0,$$

Definition of σ'

$$\partial_x u_1 + \partial_y u_2 = 0,$$

Continuity Equation

$$-\partial_x\sigma'_{11} - \partial_y\sigma'_{12} + \partial_x\hat{p} - f_1 = 0,$$
$$\partial_y\sigma'_{11} - \partial_x\sigma'_{12} + \partial_y\hat{p} - f_2 = 0,$$

Momentum Equation

where $\underline{f} = [f_1, f_2] = \hat{\rho}\underline{g}$.

Just like in the Viscosity formulation was want to add an analogous curl equation. First we must define vorticity:

$$\psi\omega - (-\partial_y u_1 + \partial_x u_2) = 0.$$

To match the form of ϕ we let

$$\psi = \frac{1}{2}({\sigma'_{11}}^2 + {\sigma'_{12}}^2 + \omega^2).$$

Using the equations for vorticity, continuity, and the definition of σ', we can construct the gradient of $\underline{u}$.

$$\nabla \underline{u} = \begin{bmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \phi\sigma'_{11} & \phi\sigma'_{12} - \psi\omega \\ \phi\sigma'_{12} + \psi\omega & -\phi\sigma'_{11} \end{bmatrix}$$

Fluidity Formulation

Stress-Gradient-Vorticity-Fluidity

We can now add $\nabla \times \nabla \underline{u} = 0$ to our system to get:

Stress-Gradient-Vorticity-Fluidity Formulation

$$\phi\sigma'_{11} - (\partial_x u_1 - \partial_y u_2) = 0,$$
$$\phi\sigma'_{12} - (\partial_y u_1 + \partial_x u_2) = 0,$$
$$\psi\omega - (-\partial_y u_1 + \partial_x u_2) = 0,$$

Definition of σ', ω

$$\partial_x u_1 + \partial_y u_2 = 0,$$

Continuity Equation

$$-\partial_x \sigma'_{11} - \partial_y \sigma'_{12} + \partial_x \hat{p} - f_1 = 0,$$
$$\partial_y \sigma'_{11} - \partial_x \sigma'_{12} + \partial_y \hat{p} - f_2 = 0,$$

Momentum Equation

$$-\partial_y(\phi\sigma'_{11}) + \partial_x(\phi\sigma'_{12}) - \partial_x(\psi\omega) = 0,$$
$$-\partial_x(\phi\sigma'_{11}) - \partial_y(\phi\sigma'_{12}) - \partial_y(\psi\omega) = 0.$$

Curl Equation

Outline:

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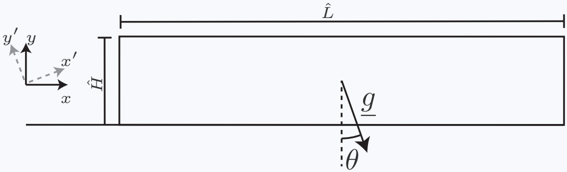
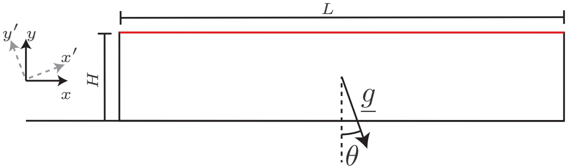
1. FOSLS Formulations
2. Test Problems
   - Rectangular Domain
   - Rec Domain - Results
   - ISMIP-HOM Benchmark B
   - BenchB - Results
3. Future Plans

Computational Domain



with $\underline{g} = |g|[\sin(\theta), -\cos(\theta)]^T$, $\hat{H} = 1$ km, and $\hat{L} = 10$ km.

Bottom	Top	Ends
$u_1 = 0$ $u_2 = 0$ $\nabla \underline{u} \cdot \underline{t} = 0$	$\left(\underline{\sigma}' - \hat{p}\underline{I}\right) \cdot \underline{n} = 0$	Periodic

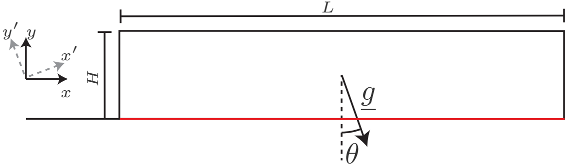


For the Top boundary we want to impose a stress free condition ($\underline{\underline{\sigma}} \cdot \underline{n} = 0$).

$$\begin{aligned}\underline{\underline{\sigma}} \cdot \underline{n} &= \left(\underline{\underline{\sigma}}' - \hat{p}\underline{\underline{I}}\right) \cdot \underline{n} \\ &= \begin{bmatrix} \sigma'_{11} - \hat{p} & \sigma'_{12} \\ \sigma'_{12} & -\sigma'_{12} - \hat{p} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} \sigma'_{12} \\ \sigma'_{11} + \hat{p} \end{bmatrix} = 0.\end{aligned}$$

Rectangular Glacier

Boundary Conditions: Bottom & Sides

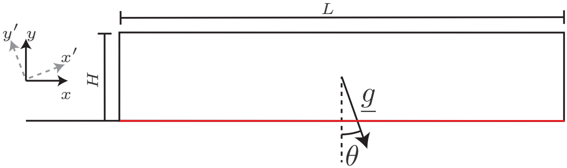


Assume the glacier is frozen to the bed (no slip)

$$\underline{u} = 0$$

This also gives us :

$$\nabla \underline{u} \cdot t = 0$$

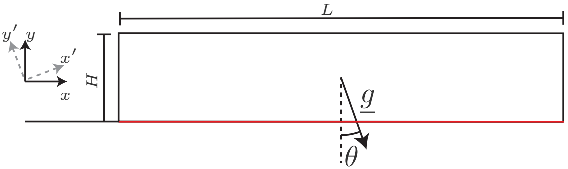


Assume the glacier is frozen to the bed (no slip)

$$\underline{u} = 0$$

This also gives us :

$$\nabla \underline{u} \cdot t = \begin{bmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

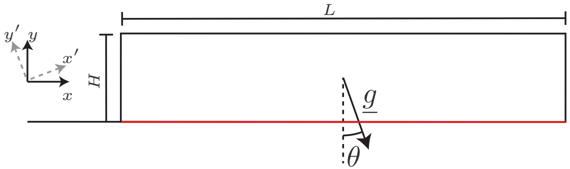


Assume the glacier is frozen to the bed (no slip)

$$\underline{u} = 0$$

This also gives us :

$$\nabla \underline{u} \cdot t = \frac{1}{2} \begin{bmatrix} \phi\sigma'_{11} & \phi\sigma'_{12} - \psi\omega \\ \phi\sigma'_{12} + \psi\omega & -\phi\sigma'_{11} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$



Assume the glacier is frozen to the bed (no slip)

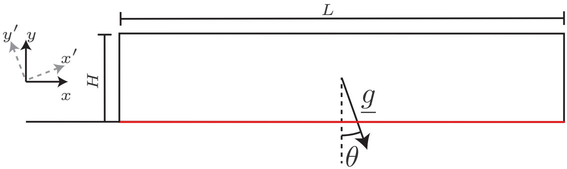
$$\underline{u} = 0$$

This also gives us :

$$\nabla \underline{u} \cdot t = \begin{bmatrix} \sigma'_{11} \\ \phi\sigma'_{12} + \psi\omega \end{bmatrix} = 0$$

Rectangular Glacier

Boundary Conditions: Bottom & Sides



Assume the glacier is frozen to the bed (no slip)

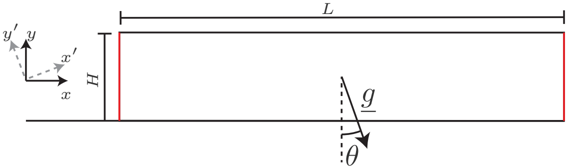
$$\underline{u} = 0$$

This also gives us :

$$\nabla \underline{u} \cdot t = \begin{bmatrix} \sigma'_{11} \\ \frac{\phi}{\psi}\sigma'_{12} + \omega \end{bmatrix} = 0$$

Rectangular Glacier

Boundary Conditions: Bottom & Sides



Assume the glacier is frozen to the bed (no slip)

$$\underline{u} = 0$$

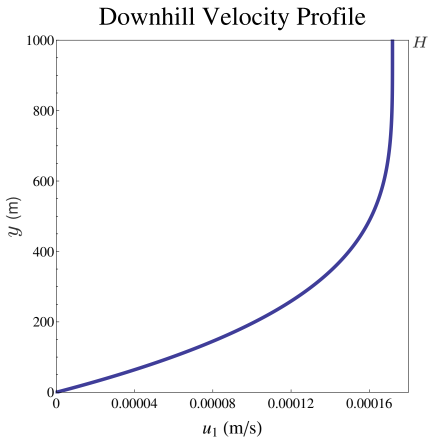
This also gives us :

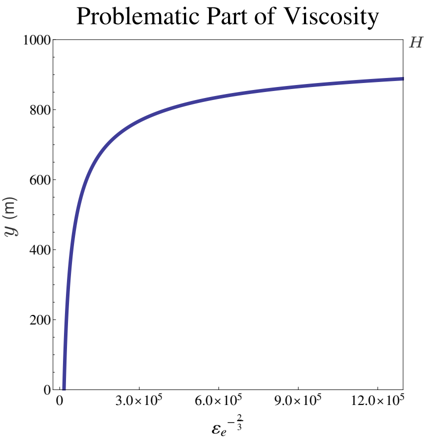
$$\nabla \underline{u} \cdot t = \begin{bmatrix} \sigma'_{11} \\ \frac{\phi}{\psi}\sigma'_{12} + \omega \end{bmatrix} = 0$$

Finally, assume periodic side boundaries.

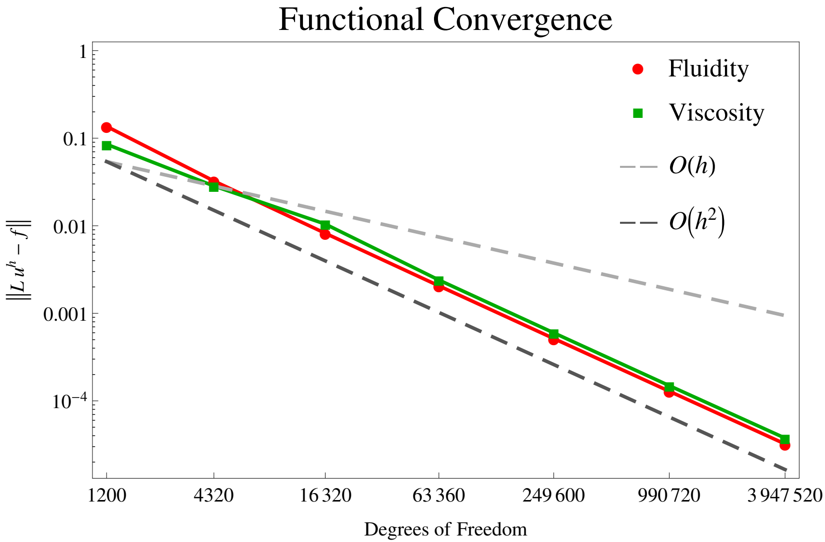
Rectangular Glacier

Exact Solution





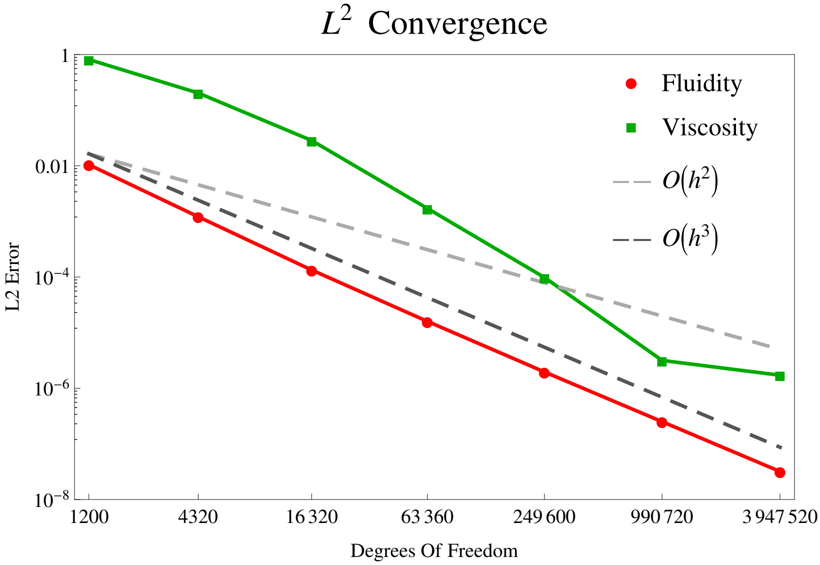
Outline:



Functional Convergence
1
0.1
0.01
0.001
10^−4
‖L u^h − f‖
Fluidity
Viscosity
O(h)
O(h^2)
1200
4320
16 320
63 360
249 600
990 720
3 947 520
Degrees of Freedom

Rectangular Glacier

L^2 Error



Rectangular Glacier

Work Units

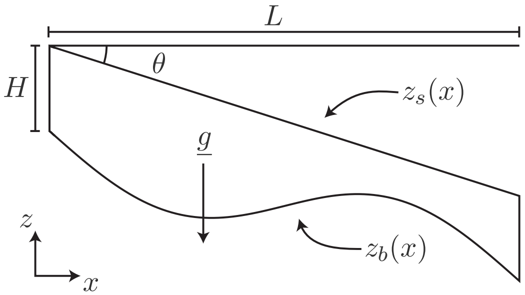
Level	E	Nonzeros	N	Comp	V-Cycles	WU	Functional
			Viscosity Formulation				
1	80	72000	22	1.57	60.82	1.49	8.52×10^{-2}
2	320	276480	4	1.78	55.25	1.07	2.85×10^{-2}
3	1280	1082880	6	1.94	96.67	11.94	1.05×10^{-2}
4	5120	4285440	10	1.97	112.90	93.57	2.40×10^{-3}
5	20480	17049600	8	2.01	129.00	346.70	5.98×10^{-4}
6	81920	68014080	6	2.03	130.30	1061.00	1.49×10^{-4}
7	327680	271687680	3	2.04	124.30	2033.00	3.76×10^{-5}
					Total	3550.00	
			Fluidity Formulation				
1	80	72000	4	1.34	13.50	0.05	1.36×10^{-1}
2	320	276480	3	1.62	14.67	0.19	3.25×10^{-2}
3	1280	1082880	3	1.78	16.67	0.94	8.21×10^{-3}
4	5120	4285440	3	1.86	17.67	4.15	2.06×10^{-3}
5	20480	17049600	2	1.91	18.00	11.49	5.15×10^{-4}
6	81920	68014080	1	1.95	17.00	22.11	1.29×10^{-4}
7	327680	271687680	1	1.96	19.00	99.34	3.22×10^{-5}
					Total	138.30	

Outline:

1. FOSLS Formulations
2. Test Problems
    - Rectangular Domain
    - Rec Domain - Results
    - ISMIP-HOM Benchmark B
    - BenchB - Results
3. Future Plans

Benchmark B

Domain

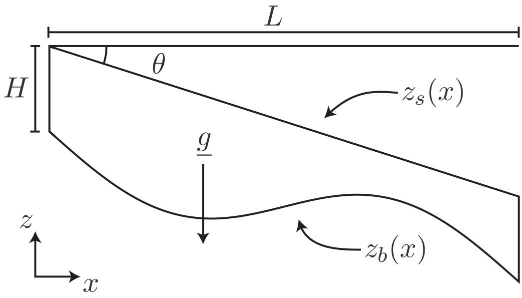


The surface of the glacier is prescribed by

$$z_s(x) = -\tan(\theta)x,$$

and the basal topography is prescribed by

$$z_b(x) = z_s(x) - H + \beta H \sin(wx).$$



Bottom	Top	Ends
$u_1 = 0$ $u_2 = 0$ $\nabla \underline{u} \cdot \underline{t} = 0$	$\left(\underline{\underline{\sigma}}' - \hat{p}\underline{\underline{I}}\right) \cdot \underline{n} = 0$	Periodic

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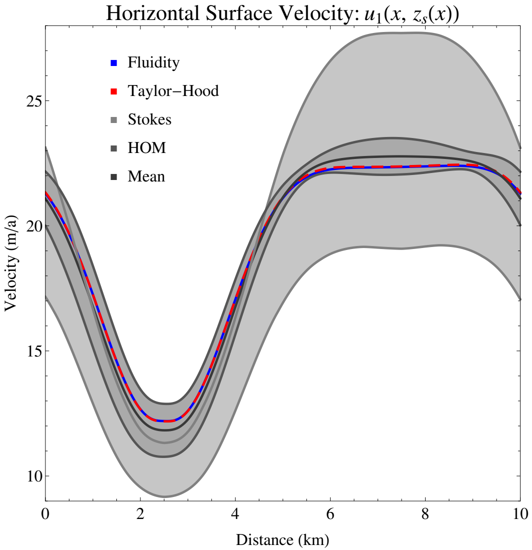
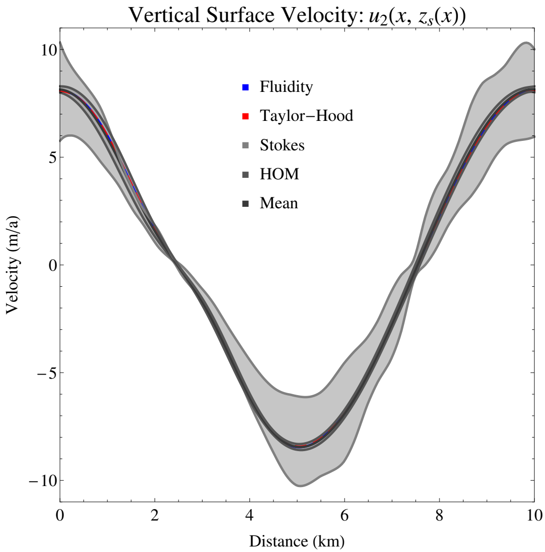
Benchmark B

Benchmark Plots



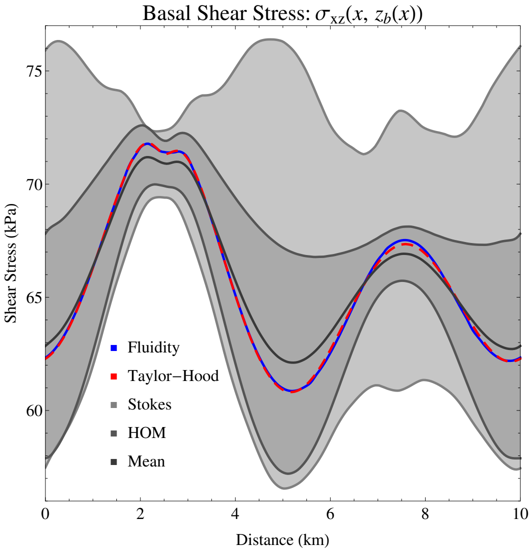
Benchmark B

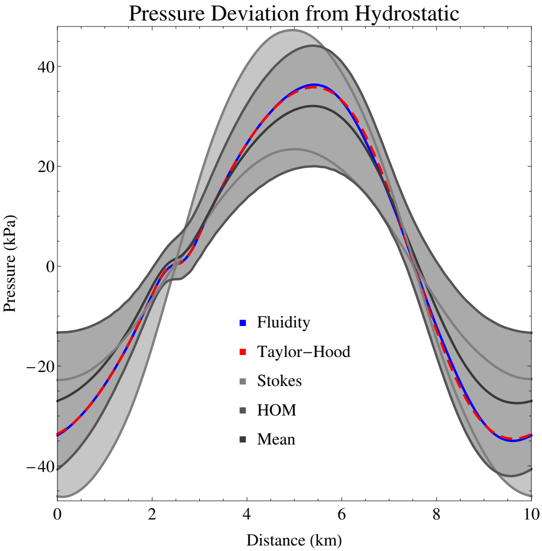
Benchmark Plots



Benchmark B

Benchmark Plots

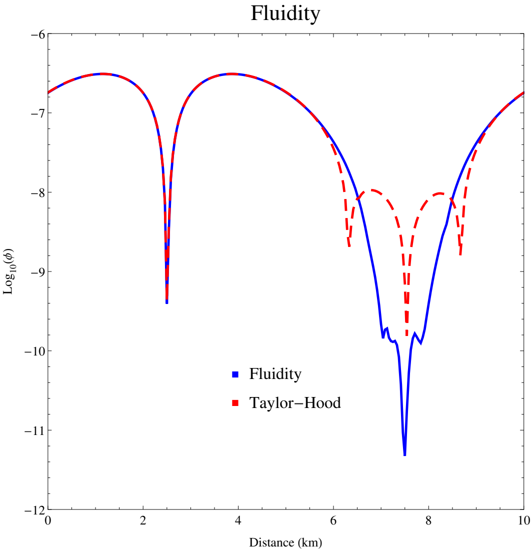




Pressure Deviation from Hydrostatic
Pressure (kPa)
40
20
0
−20
−40
0
2
4
6
8
10
Distance (km)
Fluidity
Taylor−Hood
Stokes
HOM
Mean

Benchmark B

Benchmark Plots



Outline:

1 FOSLS Formulations

2 Test Problems

3 Future Plans

Future Work

1. Mass Conservation Study
2. Other Flow Laws
3. More Efficient Iterative Solver ($H(div)$ or H^1)
4. ISMIP-HOM: Benchmark D (basal sliding)
5. Ice Shelf Modeling/Grounding Line Determination
6. Time Dependant Domains?

Questions?

Questions?

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