Calculation of Global Kinetic Energy Spectra on Irregular Grids

David Williamson

National Center for Atmospheric Research
Boulder, Colorado USA

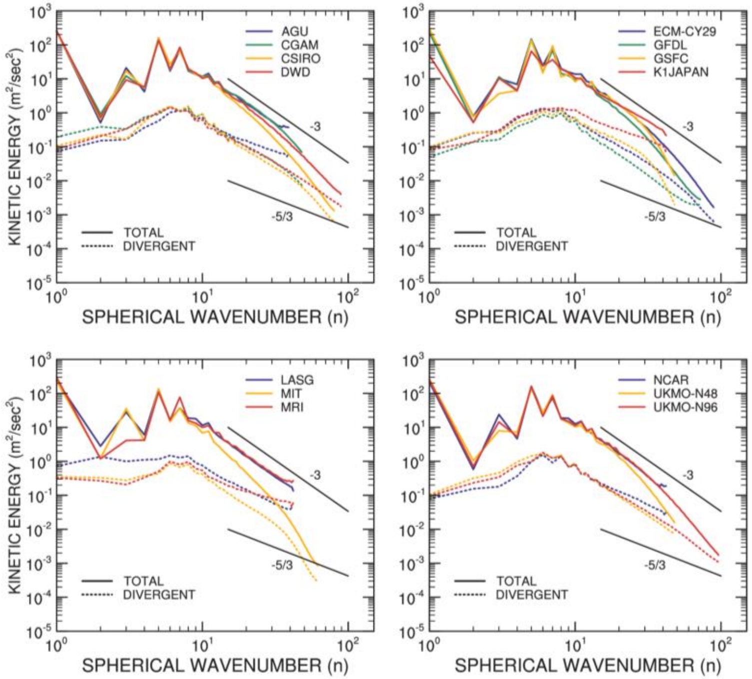


Figure 4.107: Kinetic energy spectra at 250 mb, total and divergent component (m^2 s^{-2}).

THE APE ATLAS, 2012, NCAR Technical Note, NCAR/TN-484+STR, pp. 151

$$\mathrm{KE} = \frac{1}{2\pi} \int_0^{2\pi} u^2(x)\; dx$$

$$u\,(x) = a_0 + \sum_{k=1}^{K} \left(a_k \cos kx + b_k \sin kx\right)$$

$$\mathrm{KE} = \frac{1}{2\pi} \int_0^{2\pi} \left[a_0^2 + 2a_0 \sum_{k=1}^{K} \left(a_k \cos kx + b_k \sin kx\right) \; + \left(\sum_{k=1}^{K} \left(a_k \cos kx + b_k \sin kx\right)\right)^2\right] \; dx$$

$$\mathrm{KE} = \frac{1}{2\pi} \int_0^{2\pi} u^2(x)\; dx = a_0^2 + \frac{1}{2} \sum_{k=1}^{K} \left(a_k^2 + b_k^2\right)$$

$$\int_0^{2\pi} dx = 2\pi \qquad \int_0^{2\pi} \sin^2 kx\; dx = \pi \qquad \int_0^{2\pi} \cos^2 kx\; dx = \pi, \quad k \neq 0$$

$$\int_0^{2\pi} \cos kx\; dx = 0 \qquad \int_0^{2\pi} \sin kx\; dx = 0 \qquad \int_0^{2\pi} \sin lx\;\; \cos kx\; dx = 0$$

$$\int_0^{2\pi} \sin lx\;\; \sin kx\; dx = 0 \qquad \int_0^{2\pi} \cos lx\;\; \cos kx\; dx = 0, \quad l \neq k$$

$$\mathrm{KE} = \frac{1}{2\pi} \sum_{i=1}^{N} u^2(x_i)\, \Delta x_i$$

$$u\left(x_i\right) = A_0 + \sum_{k=1}^{K} \left(A_k \cos kx_i + B_k \sin kx_i\right), \quad i = 1, 2, ..., N$$

$$\mathrm{KE} = \frac{1}{2\pi} \sum_{i=1}^{N} \left[A_0 + \sum_{k=1}^{K} \left(A_k \cos kx_i + B_k \sin kx_i\right)\right]^2 \Delta x_i$$

$$\text{UNIFORM GRID} \quad \Delta x_i = \frac{2\pi}{N}$$

$$\mathrm{KE} = \frac{1}{N} \sum_{i=1}^{N} u^2(x_i) = A_0^2 + \frac{1}{2} \sum_{k=1}^{K-1} \left(A_k^2 + B_k^2\right) + A_K^2$$

Define reference spectra from a CAM simulation

Calculate u on the non-uniform grid
from the Fourier coefficients of the reference

Interpolate to a uniform grid

Calculate spectra on uniform grid

Compare with reference spectra

Reference spectra from 0.3125° CAM integration: $k \leq 578$
truncated to $k \leq 144$: 1.25° resolution (288 points)

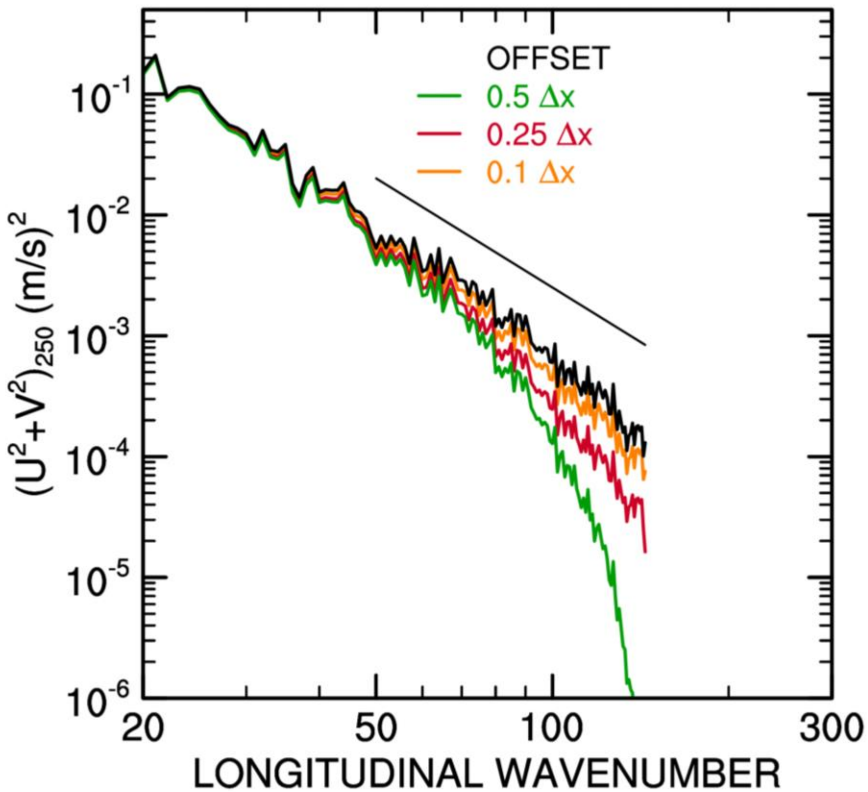
Grids considered:

- Icosahedral
- Cubed sphere, spectral element, equal angle elements
- Cubed sphere, spectral element, uniform elements on faces
- Regionally refined

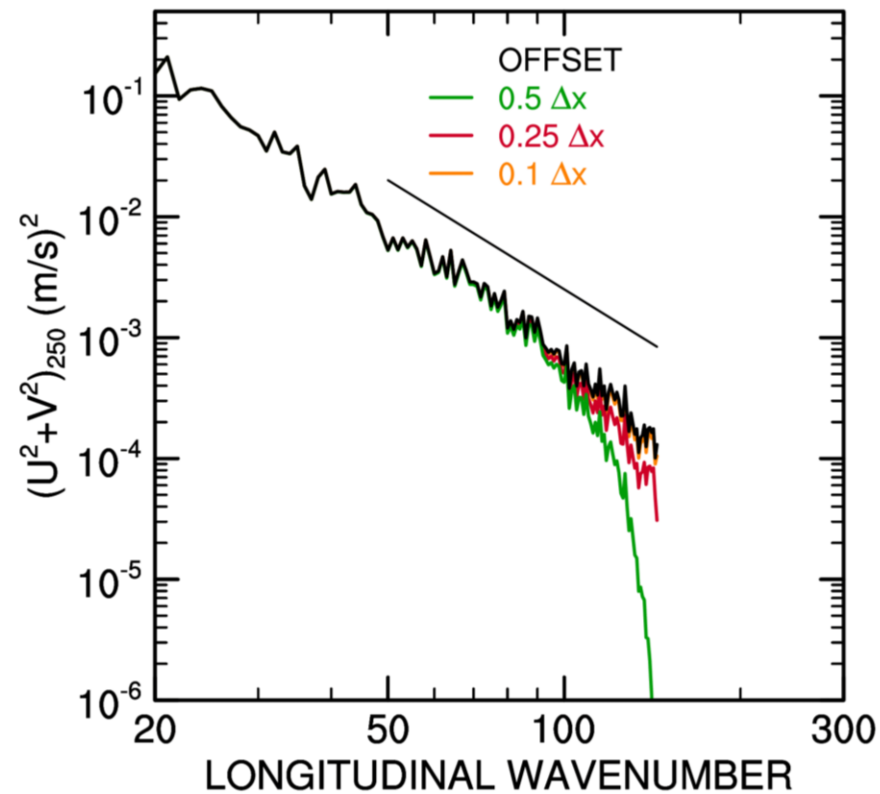
Interpolants:

- Linear
- Spline – zero tension
- Smoothing spline

LINEAR INTERPOLATION



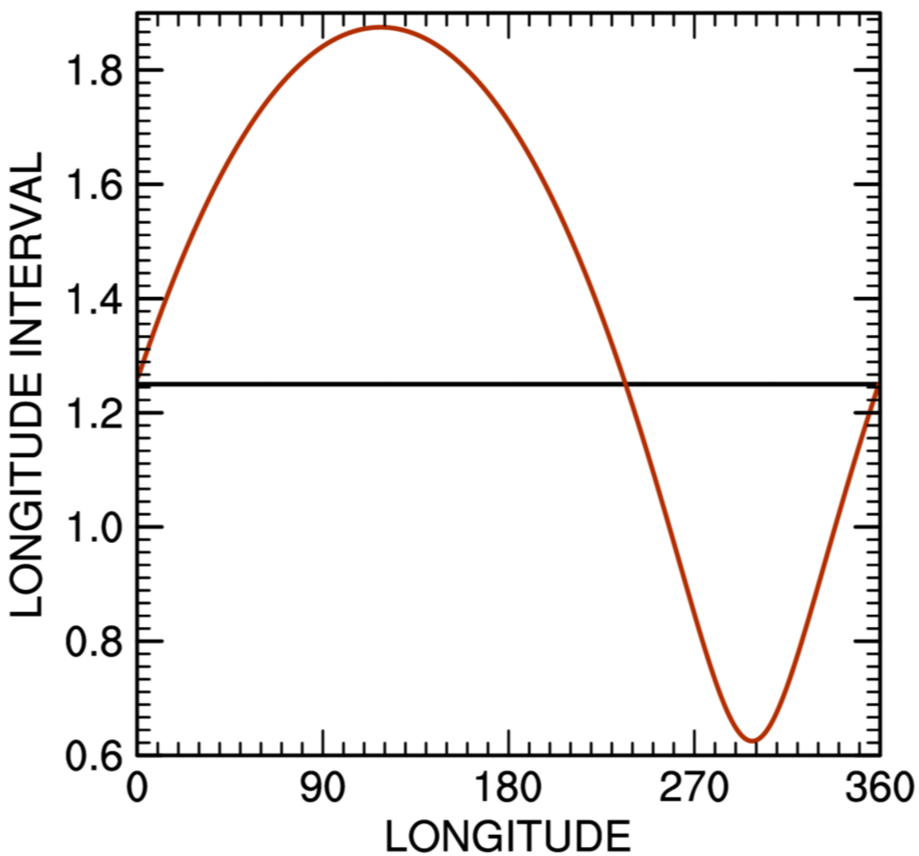
CUBIC SPLINE INTERPOLATION



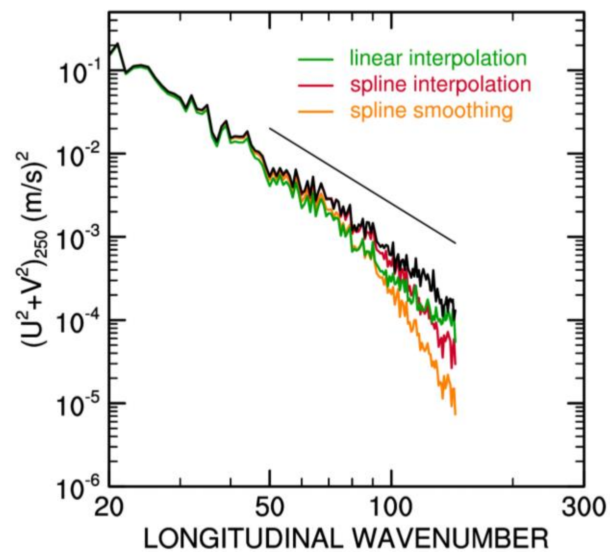
REGIONALLY REFINED GRID

$$\Delta\lambda_n = 1 + \frac{1}{2}\sin\left(n\frac{2\pi}{N}\right) \qquad n = 0, ..., N-1$$

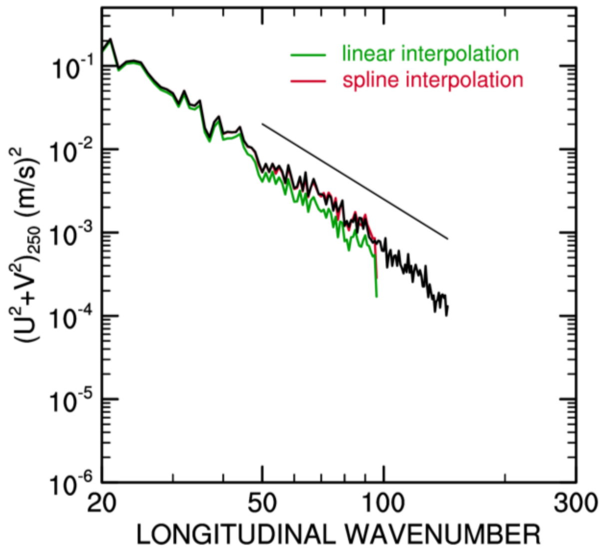
$$\lambda_n = \sum_{i=0}^{n} \Delta\lambda_n \qquad n = 0, ..., N-1$$



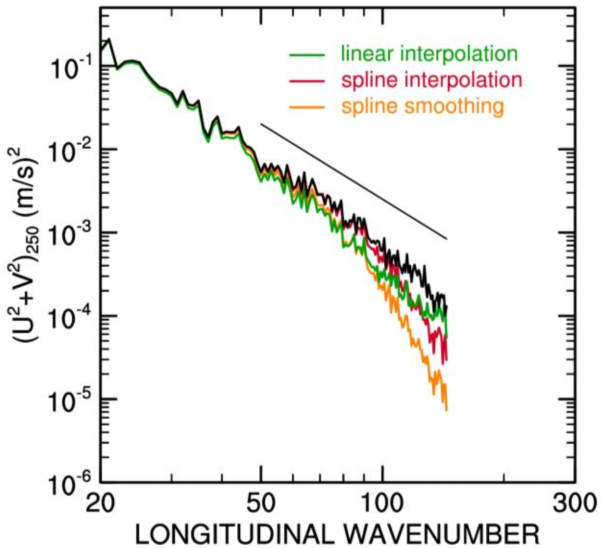
Average interval
288 points



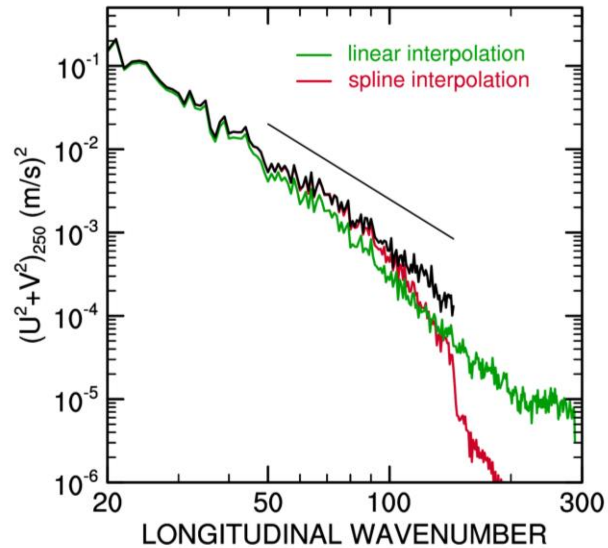
Coarsest interval
192 points



Average interval
288 points



Finest interval
576 points



LEAST-SQUARES FIT

$$u\left(x_i\right) = A_0 + \sum_{k=1}^{K} \left(A_k \cos kx_i + B_k \sin kx_i\right), \quad i = 1, 2, ..., N$$

$$\sum_{i=1}^{N} \left[u_i - A_0 - \sum_{k=1}^{K-1} \left(A_k \cos kx_i + B_k \sin kx_i\right) + A_K \cos Kx_i\right]^2 \Delta x_i = \min$$

$$A_0 \sum_{i=1}^{N} \Delta x_i + \sum_{k=1}^{K-1} \left[A_k \sum_{i=1}^{N} \cos kx_i \Delta x_i + B_k \sum_{i=1}^{N} \sin kx_i \Delta x_i\right] + A_K \sum_{i=1}^{N} \cos Kx_i \Delta x_i = \sum_{i=1}^{N} u_i \Delta x_i$$

$$A_0 \sum_{i=1}^{N} \cos lx_i \Delta x_i + \sum_{k=1}^{K-1} \left[A_k \sum_{i=1}^{N} \cos kx_i \cos lx_i \Delta x_i + B_k \sum_{i=1}^{N} \sin kx_i \cos lx_i \Delta x_i\right]$$
$$+ A_K \sum_{i=1}^{N} \cos Kx_i \cos lx_i \Delta x_i = \sum_{i=1}^{N} u_i \cos lx_i \Delta x_i \quad l = 1, 2, ..., L = K$$

$$A_0 \sum_{i=1}^{N} \sin lx_i \Delta x_i + \sum_{k=1}^{K-1} \left[A_k \sum_{i=1}^{N} \cos kx_i \sin lx_i \Delta x_i + B_k \sum_{i=1}^{N} \sin kx_i \sin lx_i \Delta x_i\right]$$
$$+ A_K \sum_{i=1}^{N} \cos Kx_i \sin lx_i \Delta x_i = \sum_{i=1}^{N} u_i \sin lx_i \Delta x_i \quad l = 1, 2, ..., L-1 = K-1$$

$$\mathbf{C}\ \mathbf{AB} = \mathbf{U}$$

$$\mathbf{AB} = (A_0\ A_1\ B_1\ \cdots\ A_k\ B_k\ \cdots\ A_{L-1}\ B_{L-1}\ A_L)^T$$

$$\begin{aligned}
C_{2l-1,2k-1} &= \sum_{i=1}^{N} \cos kx_i \cos lx_i \Delta x_i \\
C_{2l-1,2k} &= \sum_{i=1}^{N} \sin kx_i \cos lx_i \Delta x_i \\
C_{2l,2k-1} &= \sum_{i=1}^{N} \cos kx_i \sin lx_i \Delta x_i \\
C_{2l,2k} &= \sum_{i=1}^{N} \sin kx_i \sin lx_i \Delta x_i
\end{aligned}$$

$$\begin{aligned}
U_{2l-1} &= \sum_{i=1}^{N} u_i \cos lx_i \Delta x_i \\
U_{2l} &= \sum_{i=1}^{N} u_i \sin lx_i \Delta x_i
\end{aligned}$$

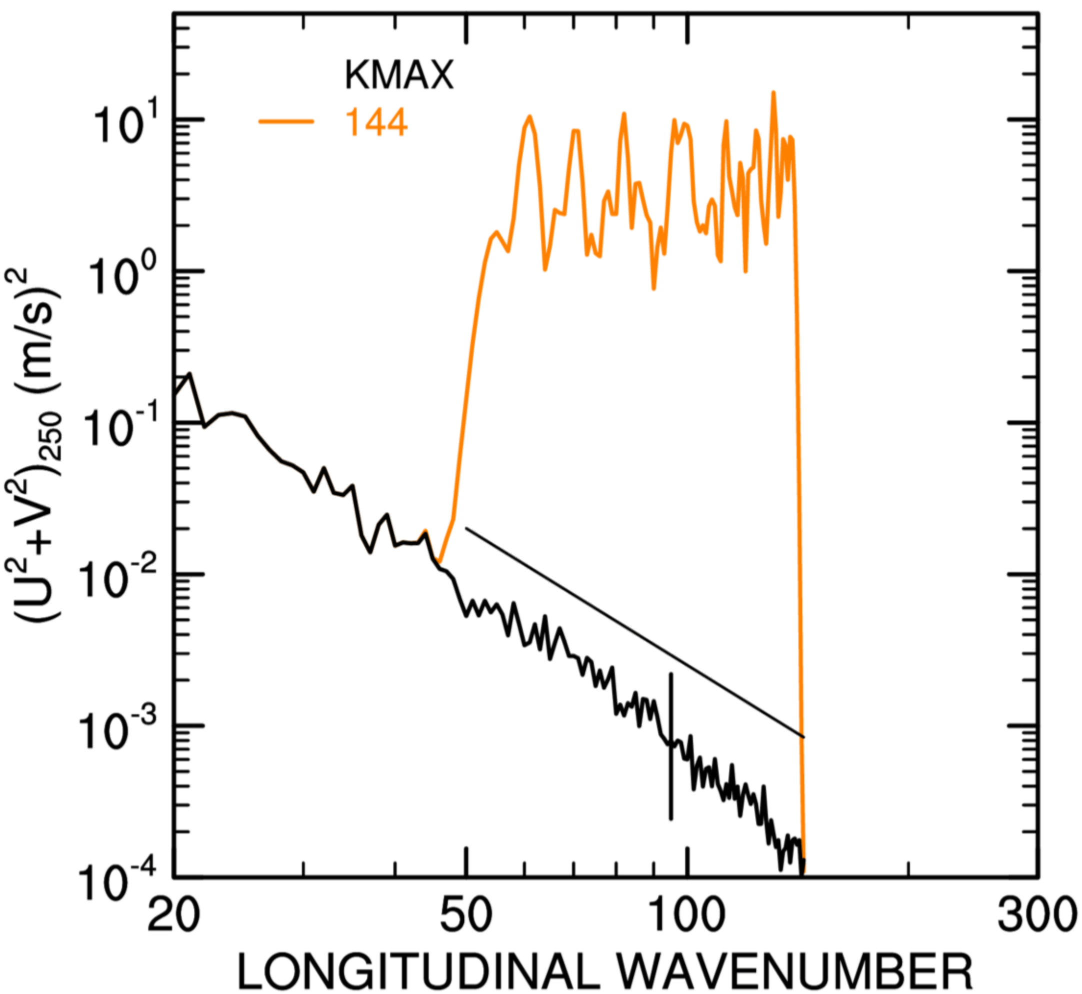
$$u\left(x_i\right) = A_0 + \sum_{k=1}^{K} \left(A_k \cos kx_i + B_k \sin kx_i\right), \quad i = 1, 2, ..., N$$

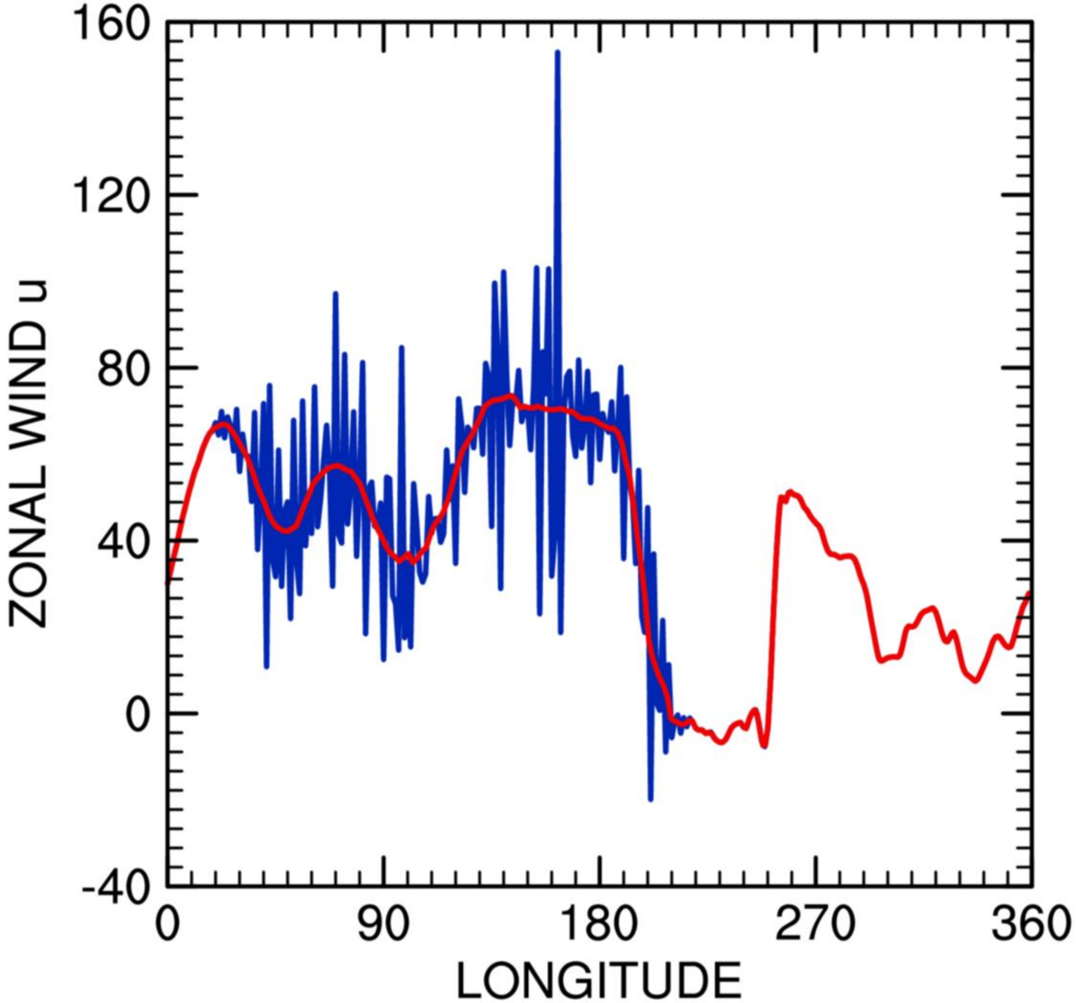
$$\mathrm{KE} = \frac{1}{2\pi} \sum_{i=1}^{N} \left[A_0 + \sum_{k=1}^{K} \left(A_k \cos kx_i + B_k \sin kx_i\right)\right]^2 \Delta x_i$$

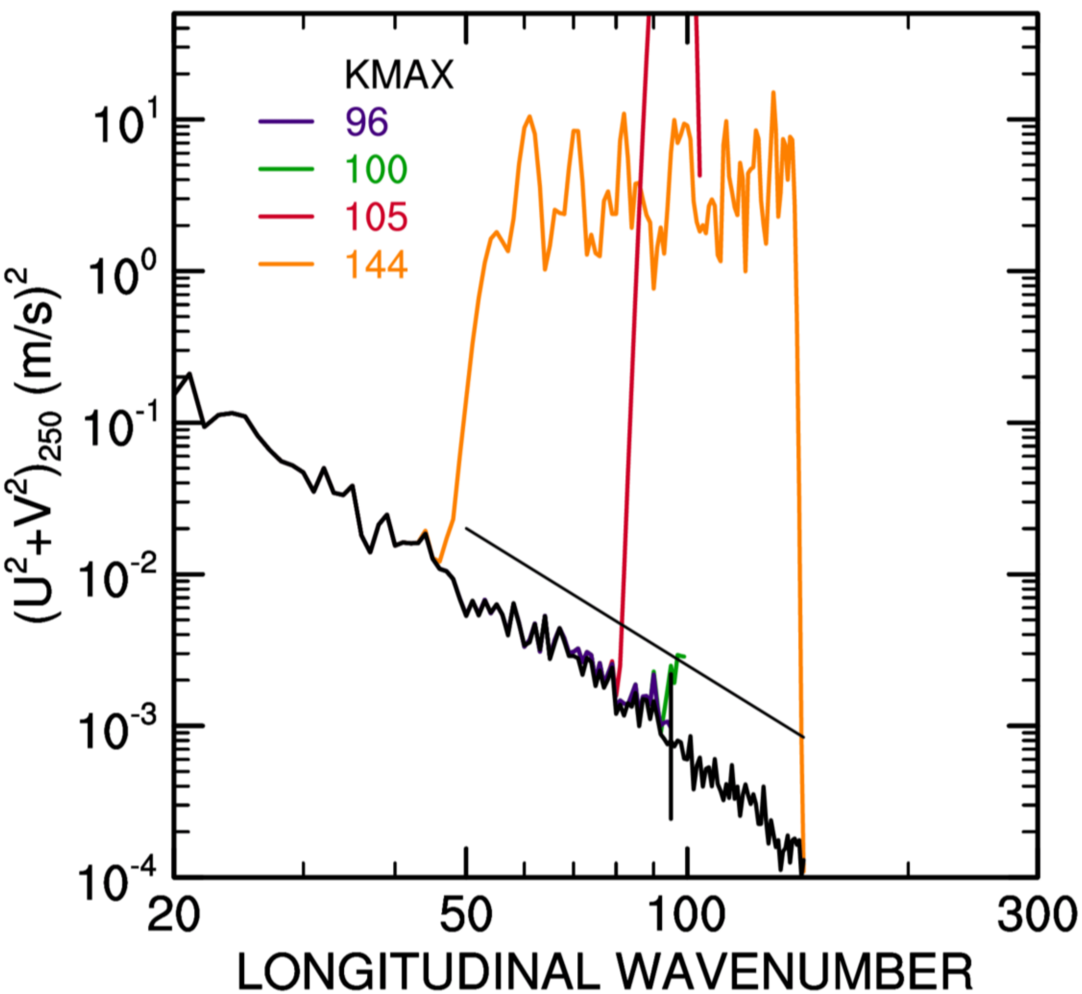
$$\mathrm{KE} = A_0^2 \frac{1}{2\pi} \sum_{i=1}^{N} \Delta x_i + \sum_{k=1}^{K} 2A_0 A_k \frac{1}{2\pi} \sum_{i=1}^{N} \cos kx_i \, \Delta x_i + \sum_{k=1}^{K} 2A_0 B_k \frac{1}{2\pi} \sum_{i=1}^{N} \sin kx_i \, \Delta x_i$$

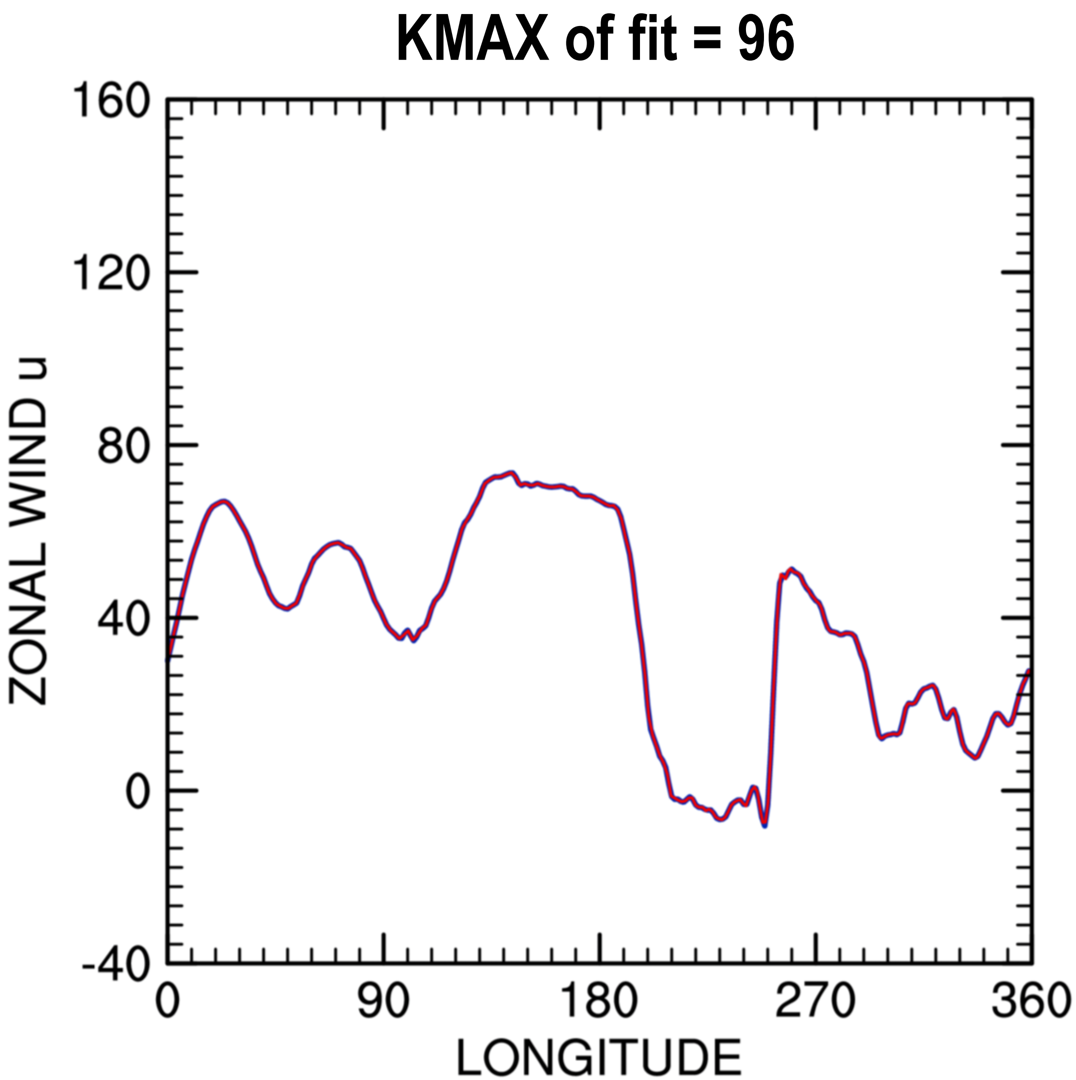
$$+ \sum_{k=1}^{K} \sum_{l=1}^{K} A_k A_l \frac{1}{2\pi} \sum_{i=1}^{N} \cos kx_i \, \cos lx_i \, \Delta x_i + \sum_{k=1}^{K} \sum_{l=1}^{K} A_k B_l \frac{1}{2\pi} \sum_{i=1}^{N} \cos kx_i \, \sin lx_i \, \Delta x_i$$

$$+ \sum_{k=1}^{K} \sum_{l=1}^{K} B_k A_l \frac{1}{2\pi} \sum_{i=1}^{N} \sin kx_i \, \cos lx_i \, \Delta x_i + \sum_{k=1}^{K} \sum_{l=1}^{K} B_k B_l \frac{1}{2\pi} \sum_{i=1}^{N} \sin kx_i \, \sin lx_i \, \Delta x_i$$


KMAX
144
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$(U^2+V^2)_{250}$ $(m/s)^2$
20
50
100
300
LONGITUDINAL WAVENUMBER


160
120
80
40
0
-40
0
90
180
270
360
ZONAL WIND u
LONGITUDE


KMAX
96
100
105
144
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
$(U^2+V^2)_{250}$ (m/s)^2
20
50
100
300
LONGITUDINAL WAVENUMBER



KMAX of fit = 96
160
120
80
40
0
-40
ZONAL WIND u
0
90
180
270
360
LONGITUDE

Best can do is calculate spectra out
to the scale of the largest grid interval

Linear interpolation introduces
an unrealistic tail-off of the spectra

Cubic spline is pretty good, but in some cases
introduces a tail-off or amplification

Least-squares fit gives correct spectra
out to scale of largest grid interval