The discrete normal modes of quasigeostrophic theory

Adapted from:

Yassin, H., in prep.: Normal Modes with Boundary Dynamics.

Yassin, H., and Griffies, S. M., in prep.: On the Discrete Normal Modes of Quasigeostrophic Theory.

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Recipe for normal modes

Recipe:

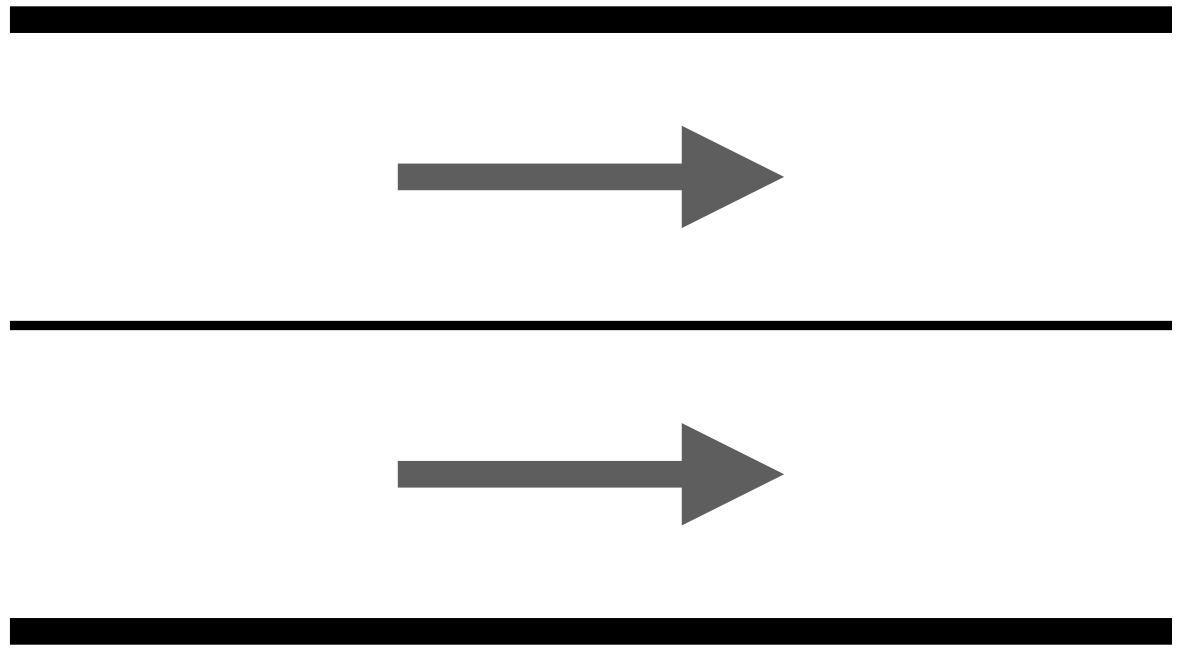
(1) Consider perturbations to a background state with no background shear.

(2) We obtain an eigenvalue problem in some space $\mathscr{F}$.

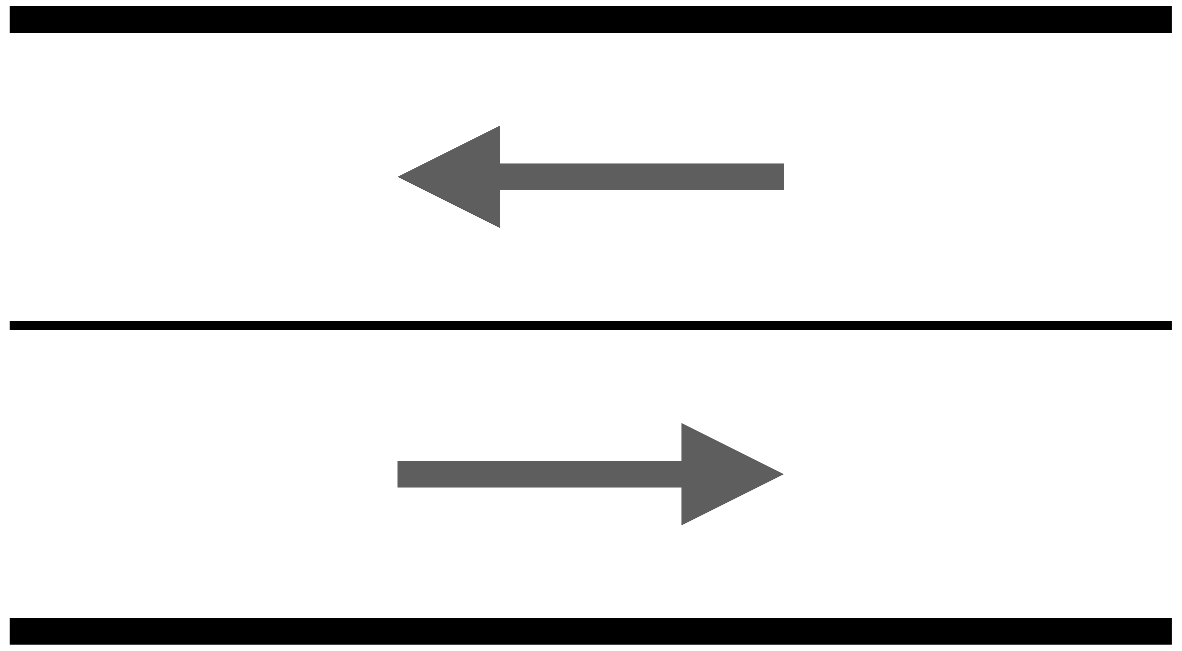
(3) The *normal modes* are the resulting vertical structures. They form a basis of $\mathscr{F}$.

In a two-layer fluid: $\mathscr{F} = \mathbb{C}^2$, (a two dimensional space).

Barotropic mode



Baroclinic mode



What mode does the altimeter signal reflect?

Vertical decomposition of QG states into "normal modes" plays an important role in physical oceanography (e.g., Charney 1971, Wunsch 1997, Smith & Vallis 2001, Lapeyre 2009).

Wunsch (1997) — Most energy is in the zeroth and first baroclinic modes.

The altimeter signal corresponds to *thermocline motion*.

Lapeyre (2009) — The baroclinic modes are unable to represent surface buoyancy and so are **incomplete**.

Altimeter signal reflects *surface-trapped* SQG motion.

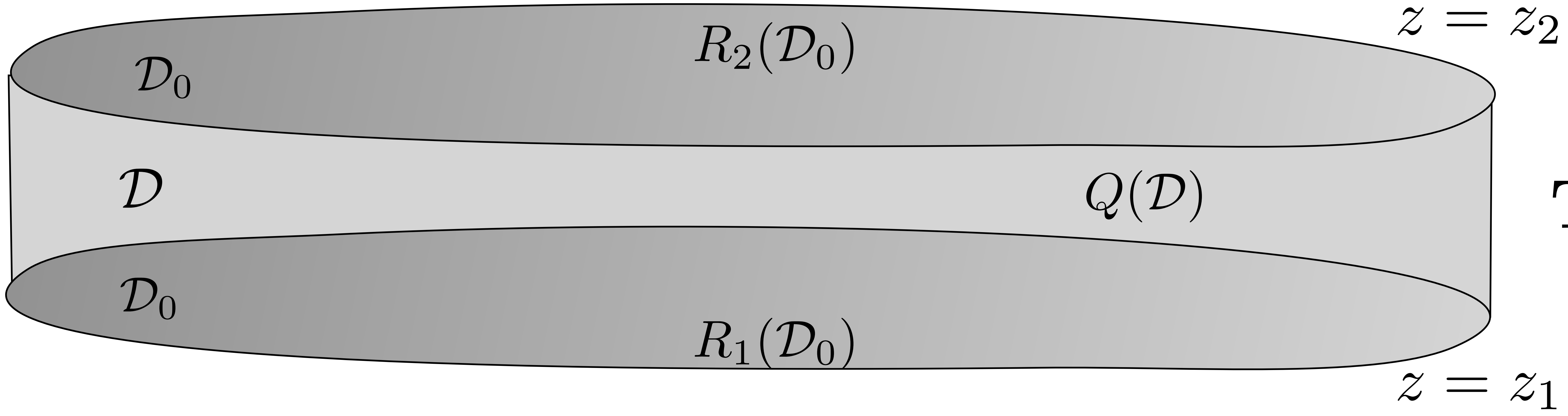
Contents

Potential Vorticity Densities

PV $(m^3 s^{-1})$

Volume PV density: $Q = f + \nabla^2\psi + \frac{\partial}{\partial z}\left[\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right]$ — $Q(x, y, z)\,\mathrm{dV}$

Surface PV density: $R_i = (-1)^{i+1}\left[f_0\, h_i + \frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right]\Bigg|_{z_i}$ for $i = 1, 2$ — $R_i(x, y)\,\mathrm{dA}$



Total PV: $\int Q\,\mathrm{dV} + \sum_{i=1}^{2}\int R_i\,\mathrm{dA}$

The Quasigeostrophic Phase Space

PV distribution: $\mathfrak{Q}(x, y, z, t) = Q(x, y, z, t) + \sum_{i=1}^{2} R_i(x, y, t)\,\delta(z - z_i)$

(Bretherton 1966)

QG Phase Space: The space of all possible QG states.

PV perspective: The space of all distributions $\mathfrak{Q}$.

Streamfunction perspective: The space of all functions ψ induced by $\mathfrak{Q}$.

Vertical Structure Phase Space

In a doubly periodic domain: $\mathfrak{Q}(x, y, z, t) = \mathfrak{Q}_{\boldsymbol{k}}(z, t)\, e^{i\left(k_1 x + k_2 y\right)}$

The vertical structure of the PV distribution is

$$\mathfrak{Q}_{\boldsymbol{k}}(z, t) = Q_{\boldsymbol{k}}(z, t) + \sum_{i=1}^{2} R_{i\,\boldsymbol{k}}(t)\, \delta(z - z_i)\,.$$

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$Q_{\boldsymbol{k}}(z)$ is an element of L^2 — where L^2 is the space of square-integrable functions on $[z_1, z_2]$

$R_{i\,\boldsymbol{k}}$ is an element of $\mathbb{C}$ — where $\mathbb{C}$ is the space complex number.

The space of all possible vertical structures in QG is

$$L^2 \oplus \mathbb{C} \oplus \mathbb{C}\,.$$

Contents

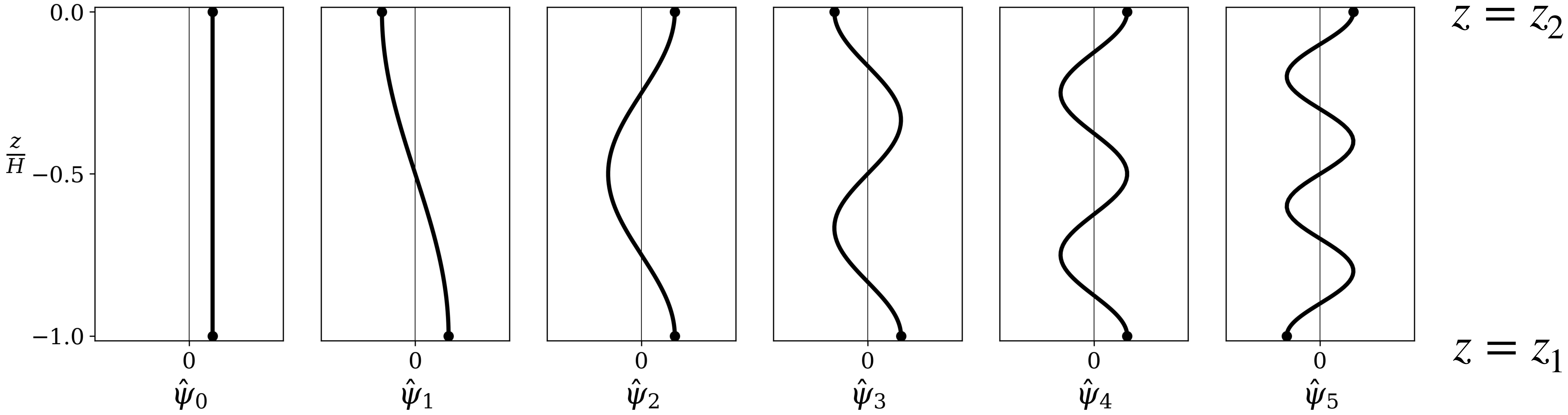
The Baroclinic Modes

Vertical structure of Rossby waves in a quiescent ocean with no topography.

Streamfunction baroclinic modes: $$-\frac{d}{dz}\left(\frac{f_0^2}{N^2}\frac{d\psi}{dz}\right) = \lambda\,\psi \qquad \text{for} \qquad z \in (z_1, z_2)$$

(Assuming $\omega \neq 0$)

$$\omega = -\frac{\beta\,k_1}{k^2+\lambda} \qquad\qquad \frac{d\psi(z_i)}{dz} = 0 \qquad \text{for} \qquad i = 1,2.$$



The Baroclinic Modes

Vertical structure of Rossby waves in a quiescent ocean with no topography.

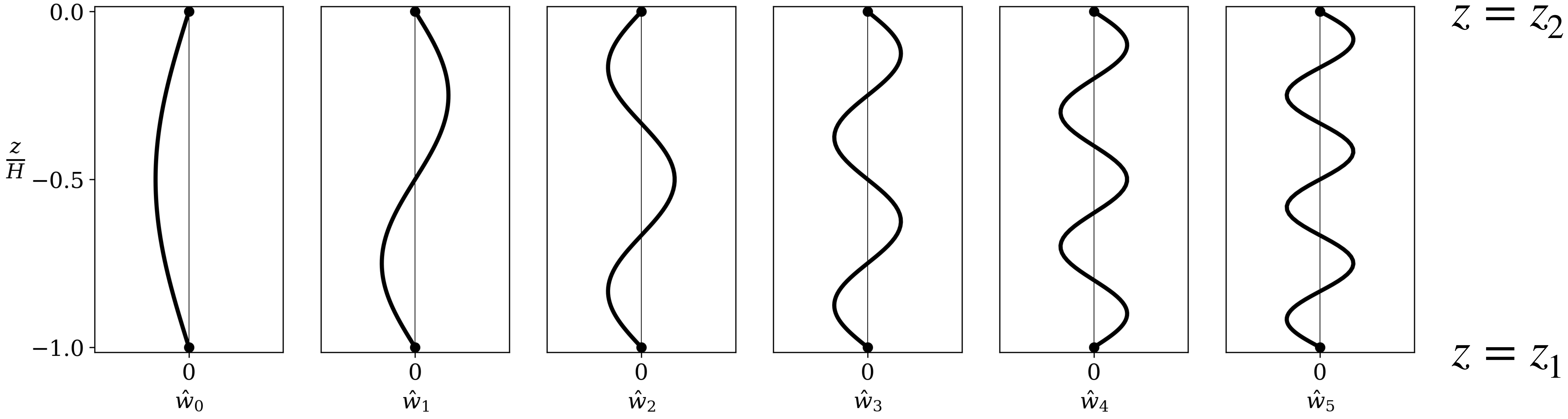
Vertical velocity baroclinic modes:

(Assuming $\omega \neq 0$)

$$\omega = -\frac{\beta k_1}{k^2 + \lambda}$$

$$-\frac{d^2 w}{dz^2} = \lambda \left(\frac{N^2}{f_0^2}\right) w \quad \text{for} \quad z \in (z_1, z_2)$$

$$w(z_i) = 0 \quad \text{for} \quad i = 1,2.$$



The modes $\{\hat{\psi}_n\}_{n=0}^{\infty}$ and $\{\hat{w}_n\}_{n=0}^{\infty}$ form an orthonormal basis of L^2.

But the space of all QG vertical structures is $L^2 \oplus \mathbb{C} \oplus \mathbb{C}$.

So we are missing two degrees of freedom

General Linear Solution

Given a vertical structure $\Psi(z)$ with $d\Psi(z_i)/dz = 0$, then

$$\psi(z,t) = \sum_{n=0}^{\infty} \langle \Psi, \hat{\psi}_n \rangle \underbrace{\hat{\psi}_n(z)\, e^{-i\omega_n t}}_{\text{Physical normal mode}}$$

Assumptions:	*Infinitesimal* (linear) perturbations to a *quiescent* ocean.
	No buoyancy gradients at the boundaries.
	No topography.

Baroclinic Modes as a Basis

We can instead *try* to use the baroclinic modes, $\{\hat{\psi}_n\}_{n=0}^{\infty}$ and $\{\hat{w}_n\}_{n=0}^{\infty}$, as a bases for QG theory.

Suppose we have a QG state with $\Psi(z,t)$ and $W(z,t)$. Define

$$\Psi^{\exp}(z,t) = \sum_{n=0}^{\infty} \Psi_n(t)\,\hat{\psi}_n(z) \qquad \text{and} \qquad W^{\exp}(z,t) = \sum_{n=0}^{\infty} W_n(t)\,\hat{w}_n(z)\,.$$

When do we have $\Psi = \Psi^{\exp}$ and $W = W^{\exp}$?

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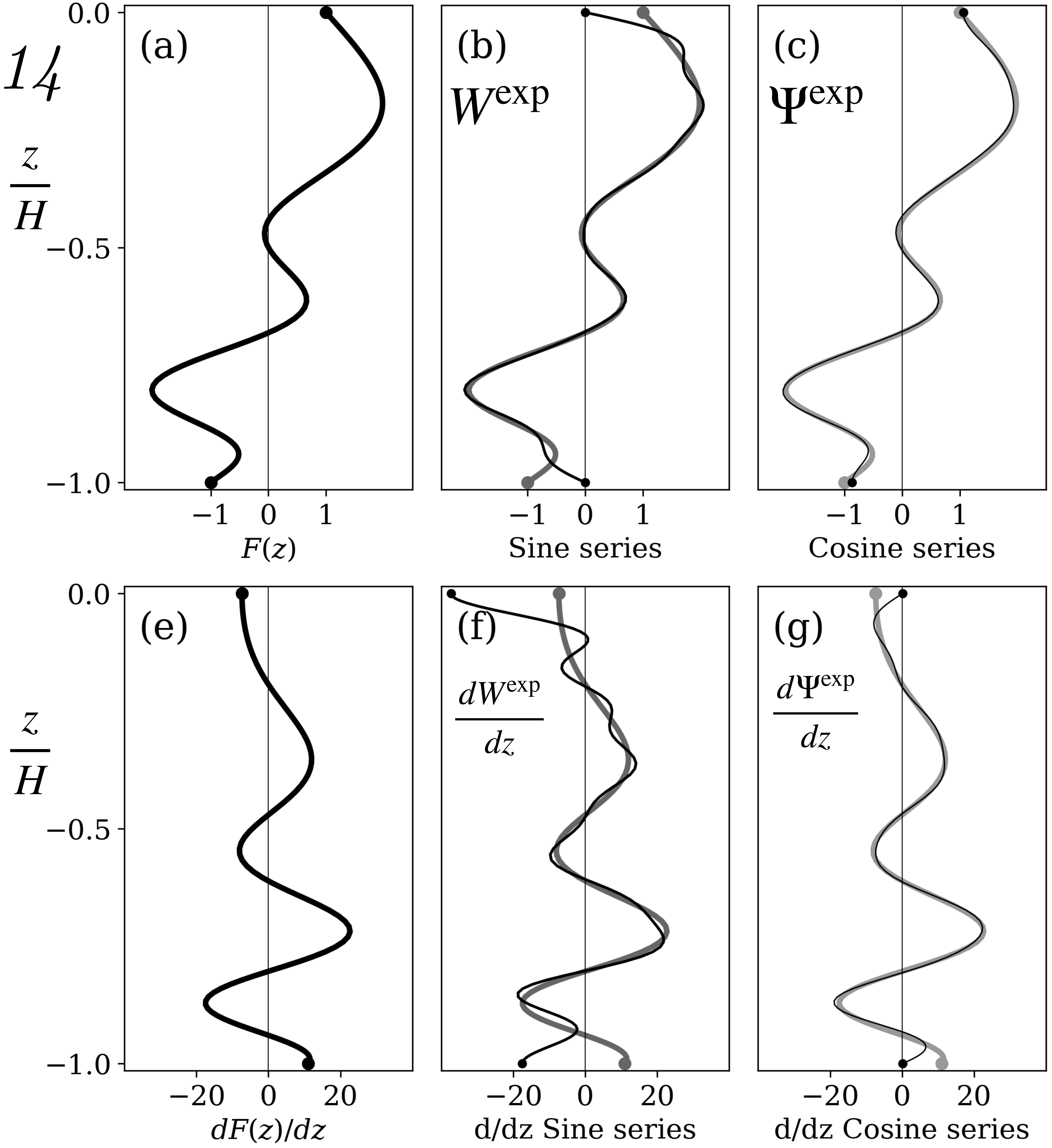


Vertical velocity — Sine-like

If $W(z_i) = 0$

then $W = W^{\mathrm{exp}}$

and $\frac{dW}{dz} = \frac{dW^{\mathrm{exp}}}{dz}$.

Streamfunction — Cosine-like

$\Psi = \Psi^{\mathrm{exp}}$ regardless of boundary conditions.

However, we only obtain $\frac{d\Psi}{dz} = \frac{d\Psi^{\mathrm{exp}}}{dz}$

if $\frac{d\Psi(z_i)}{dz} = 0$.

Expansions and the missing degrees of freedom

Given an arbitrary vertical structure $\Psi(z)$, we have

$$\Psi(z) = \sum_{n=0}^{\infty} \Psi_n \hat{\psi}_n(z) .$$

Volume potential vorticity: Differentiate the series in the *interior* for $z \in (z_1, z_2)$.

$$Q(z) = -\sum_{n=0}^{\infty} \left(k^2 + \lambda_n\right) \Psi_n \hat{\psi}_n(z) .$$

The series is not differentiable at $z = z_1, z_2$, hence we have *lost* the surface potential vorticities R_1 and R_2 in the projection process.

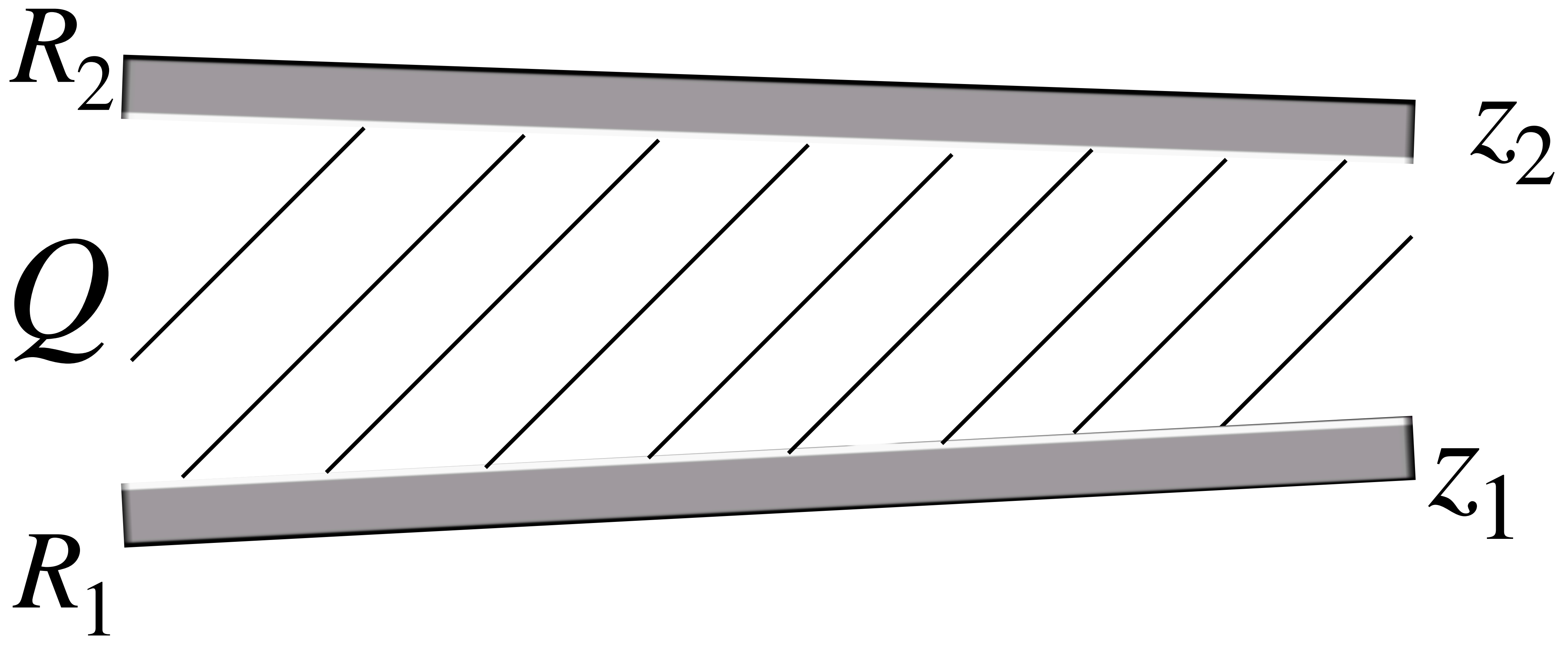
Contents

The Generalized Rhines Modes

Vertical structure of Rossby waves in a quiescent ocean with prescribed boundary buoyancy gradients / topography.

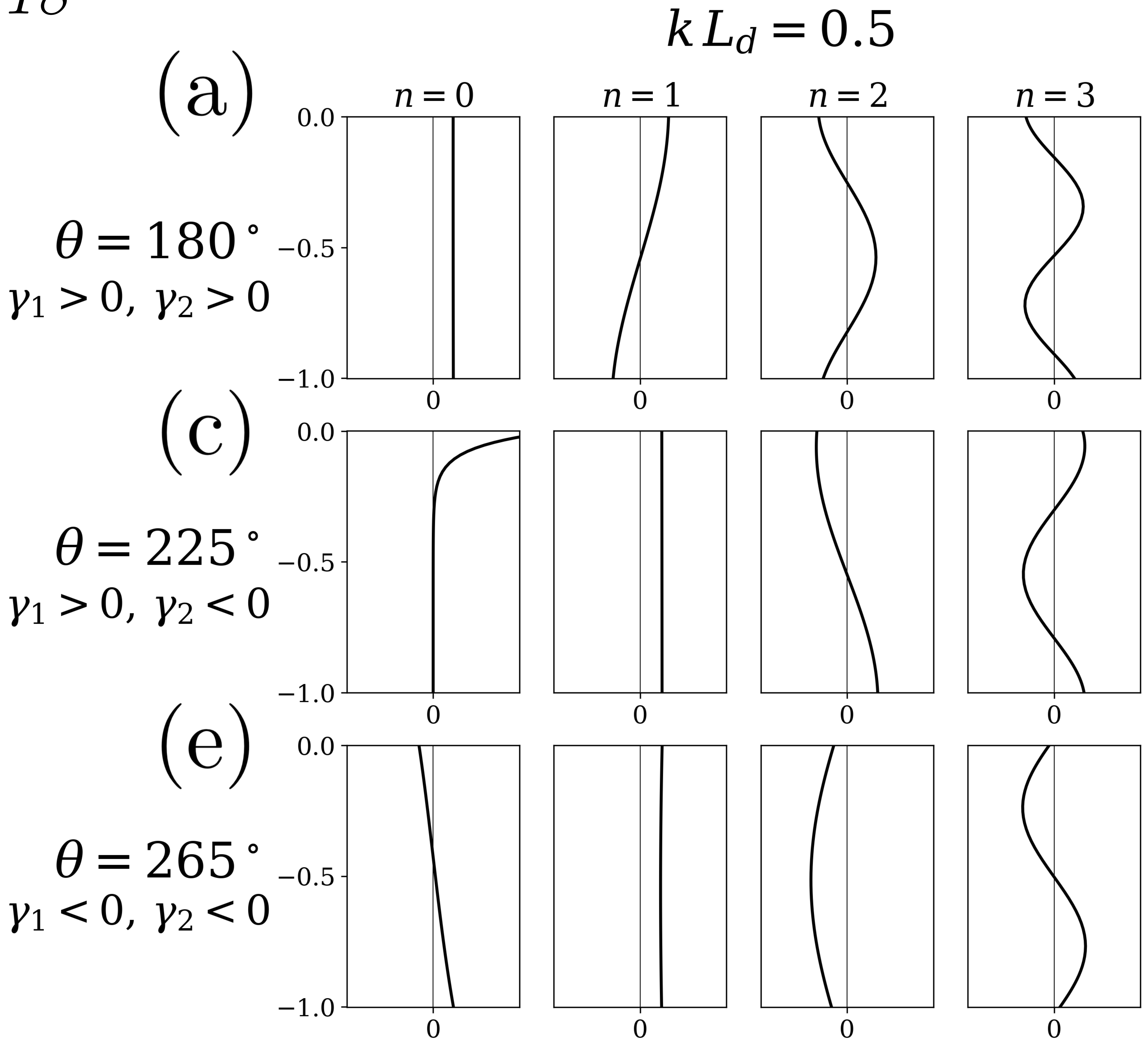
$$-\frac{d}{dz}\left(\frac{f_0^2}{N^2}\frac{d\psi}{dz}\right) = \lambda\,\psi \qquad \text{for} \quad z \in (z_1, z_2)$$

$$-k^2\psi + (-1)^i\,\gamma_i^{-1}\left(\frac{f_0^2}{N^2}\frac{d\psi}{dz}\right) = \lambda\,\psi \qquad \text{at} \quad z = z_i$$



$$\gamma_i = (-1)^{i+1}\frac{\hat{\boldsymbol{z}}\cdot\left(\boldsymbol{k}\times\nabla g_i\right)}{\hat{\boldsymbol{z}}\cdot\left(\boldsymbol{k}\times\nabla f\right)} = (-1)^{i+1}\frac{|\nabla g_j|\,k}{\beta\,k_x}\,\sin\left(\Delta\theta_i\right)$$

where $g_i = f_0\,h_i$ for topography.



(*Large* horizontal scales)

The nth mode has n zeros.

Two modes ($n = 0{,}1$) with no zeros.

Two modes with one zero ($n = 0{,}3$).

Two modes with no zeros ($n = 1{,}2$).

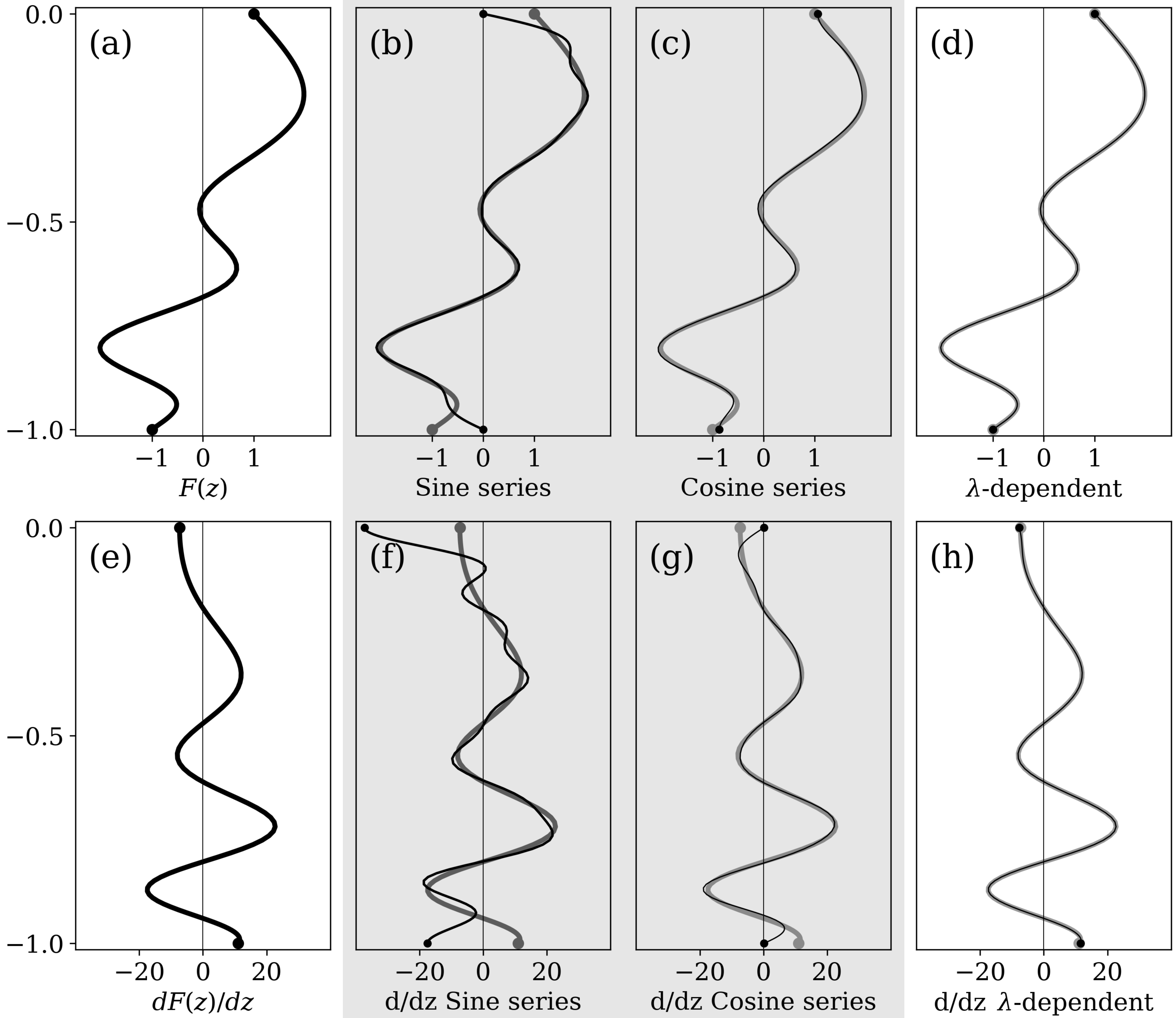
Expansion Properties

The generalized Rhines modes $\{\hat{\psi}_n\}_{n=0}^{\infty}$ form an orthonormal basis of $L^2 \oplus \mathbb{C} \oplus \mathbb{C}$.

Given a QG state with streamfunction $\Psi(z)$, then

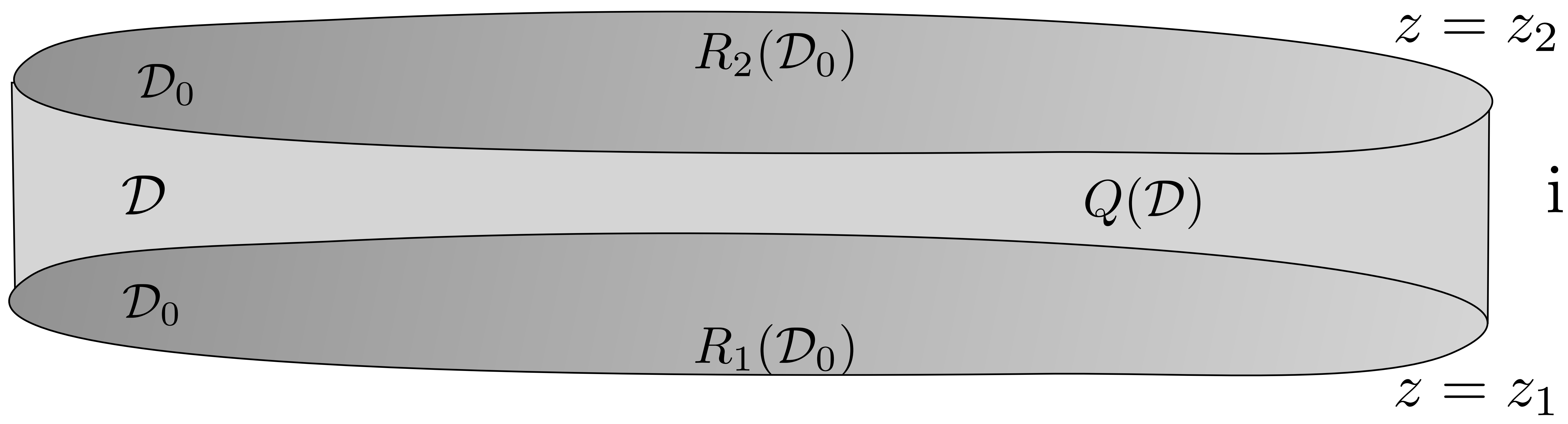
$$\Psi(z) = \sum_{n=0}^{\infty} \Psi_n \hat{\psi}_n(z) \qquad \text{and} \qquad \frac{d\Psi(z)}{dz} = \sum_{n=0}^{\infty} \Psi_n \frac{d\hat{\psi}_n(z)}{dz} \qquad \text{on} \quad z \in [z_1, z_2]$$

regardless of what boundary conditions $\Psi(z)$ *satisfies.*


(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)
0.0
−0.5
−1.0
−1
0
1
−20
0
20
F(z)
Sine series
Cosine series
λ-dependent
dF(z)/dz
d/dz Sine series
d/dz Cosine series
d/dz λ-dependent

The misleading nature of the baroclinic modes

Consider a realistic ocean with strong surface QG dynamics and weak interior dynamics.



i.e., large $\boldsymbol{R_2}$ and small $\boldsymbol{Q}$

A projection onto the baroclinic modes (as in Wunsch 1997) may give the *misleading impression* that the motion is all due to the volume PV $\boldsymbol{Q}$.

Summary

1. The baroclinic modes are *incomplete*. Information is *lost* in the projection process.

2. There are infinitely many *complete* modes for QG theory. No physical reason to prefer one set over another.

3. In a non-linear ocean, modal decompositions do not provide any additional *physical* insight and can be *highly misleading*.

4. The question of "what mode the altimeter signal reflects" is *ill-posed.* There is no unique decomposition into normal modes in non-linear theory.