Implicit discretizations for grounding line dynamics

Jed Brown

Laboratory of Hydrology, Hydraulics, and Glaciology
ETH Zürich

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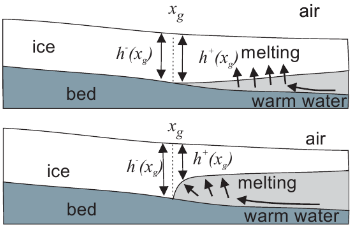
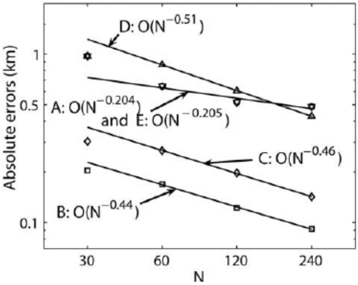
Grounding lines

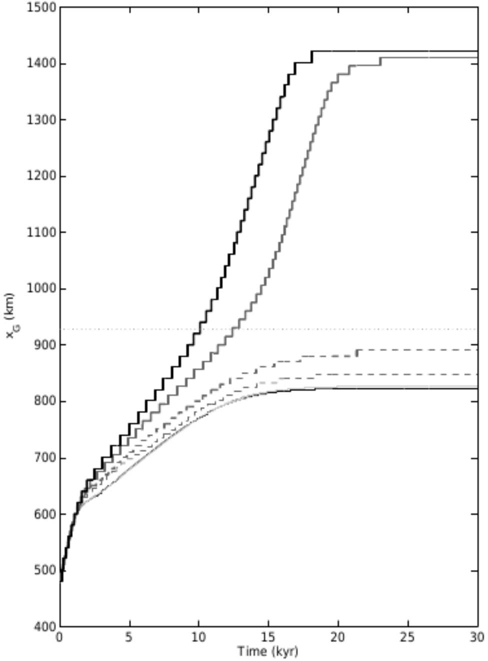


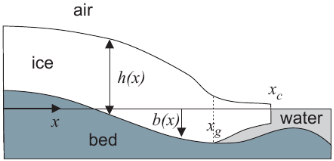
Schoof 2007



Bueler et. al. 2005

- ocean circulation is very sensitive to grounding line geometry, feedback
- non-shallow physics applies in vicinity of grounding line
- current models are less than first-order accurate at margins
- extremely high resolution needed for qualitatively correct results on Eulerian meshes

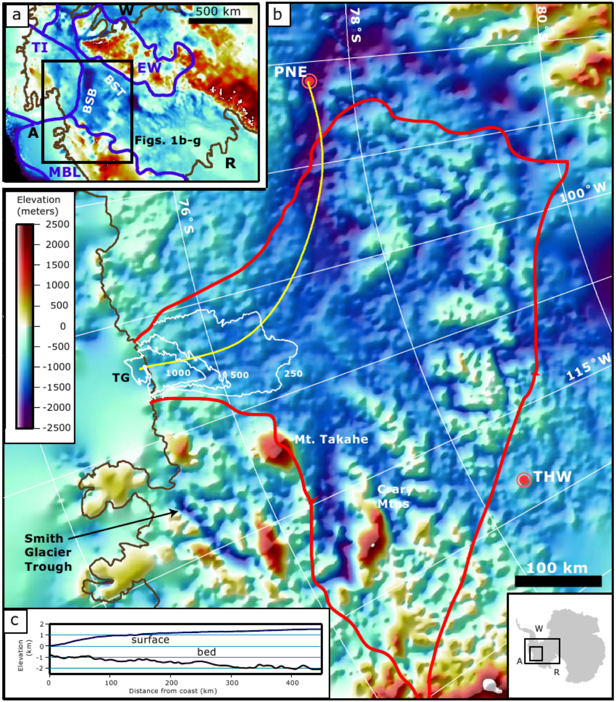




(Schoof 2007)

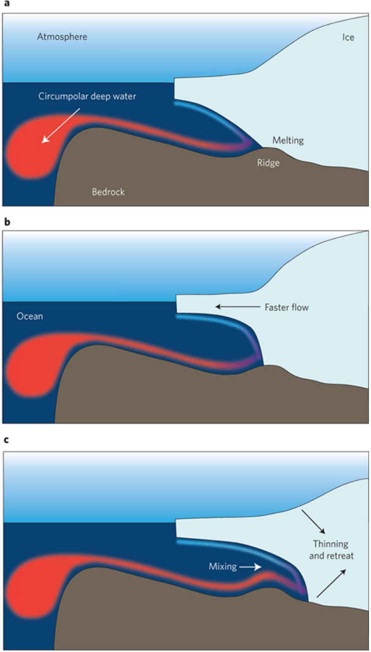
Evolution of grounding line location on 20, 15, 10, 7.5 and 2.5 kilometer meshes in one horizontal dimension.

(*Durand et al. 2009*)



Holt et al.

2006



(Schoof 2010)

y^+ underneath an ice shelf

- Order of magnitude dimensions: length $100\,\mathrm{m}$, speed $10\,\mathrm{cm/s}$
- Viscous boundary layer: $y^+ \in \mathcal{O}(1) \implies 1\,\mathrm{mm}$ grid
- No-slip boundary conditions requires *resolution* of this layer
- Otherwise we need nonlinear slip
  - still usually $y^+ \in \mathcal{O}(100)$
- Estimates come from validation (lab experiments) with heat transfer in industrial and aerospace applications
- Thermohaline boundary layer: 1–10 m
- Boundary layer equations require solution of a Riemann problem

LES+RANS with wall modeling

- State of the art for high-Reynolds separating flows
- Subshelf circulation separates when it reaches neutral buoyancy (this is a crucial limiting process)
- Is it possible to accurately predict heat transfer, separation, and overturning with $y^+ \in \mathcal{O}(10^5)$?

> *It has been repeatedly observed, especially at high Reynolds numbers and coarse grids and with the interface location being around $y^+ = \mathcal{O}(100-200)$, that the high turbulent viscosity generated by the turbulunce model in the inner region extends, as subgrid-scale viscosity, deeply into the outer LES region, causing severe damping in the resolved motion and a misrepresentation of the resolved structure as well as the time-mean properties.*

(Tessicini, Li, Leschziner, *Simulation of Separation from Curved Surfaces with Combined LES and RANS Schemes*, 2007)

Non-Newtonian Stokes system: velocity u, pressure p

$$-\nabla\cdot(\eta Du)+\nabla p-f=0$$
$$\nabla\cdot u=0$$

$$\begin{aligned} Du &= \tfrac{1}{2}\left(\nabla u+(\nabla u)^T\right) \\ \gamma(Du) &= \tfrac{1}{2}Du:Du \\ \eta(\gamma) &= B(\Theta,\dots)(\varepsilon+\gamma)^{\frac{\mathfrak{p}-2}{2}} \\ \mathfrak{p} &= 1+\tfrac{1}{\mathfrak{n}}\approx\tfrac{4}{3} \\ T &= 1-n\otimes n \end{aligned}$$

with boundary conditions

$$(\eta Du-p1)\cdot n=\begin{cases}0 & \text{free surface}\\ -\rho_w zn & \text{ice-ocean interface}\end{cases}$$

$$u=0 \qquad \text{frozen bed}, \Theta<\Theta_0$$

$$\left.\begin{aligned} u\cdot n &= g_{\text{melt}}(Tu,\dots) \\ T(\eta Du-p1)\cdot n &= g_{\text{slip}}(Tu,\dots)\end{aligned}\right\}\text{nonlinear slip}, \Theta\geq\Theta_0$$

$$g_{\text{slip}}(Tu)=\beta_{\mathfrak{m}}(\dots)|Tu|^{\mathfrak{m}-1}Tu$$

Navier $\mathfrak{m}=1$, Weertman $\mathfrak{m}\approx\frac{1}{3}$, Coulomb $\mathfrak{m}=0$.

Other critical equations

- Mesh motion: x

$$-\nabla\cdot\boldsymbol{\sigma} = 0 \qquad \boldsymbol{\sigma} = \mu\left[2Dw + (\nabla w)^T\nabla w\right] + \lambda|\nabla w|1$$

$$\text{surface: } (\dot{x} - u)\cdot n = q_{BL},\ T\boldsymbol{\sigma}\cdot n = 0 \qquad w = x - x_0$$

- Heat transport: Θ (enthalpy)

$$\frac{\partial}{\partial t}\Theta + (u - \dot{x})\cdot\nabla\Theta$$
$$-\nabla\cdot\left[\kappa_T(\Theta)\nabla T(\Theta) + \kappa_\omega\nabla\omega(\Theta) + q_D(\Theta)\right] - \eta Du : Du = 0$$

  - ALE advection
  - Thermal diffusion
  - Moisture diffusion/Darcy flow
  - Strain heating

Note: $\kappa(\Theta)$ and $q_D(\Theta)$ are very sensitive near $\Theta = \Theta_0$

Summary of primal variables in DAE

u	velocity	algebraic
p	pressure	algebraic
x	mesh location	algebraic in domain, differential at surface
Θ	enthalpy	differential

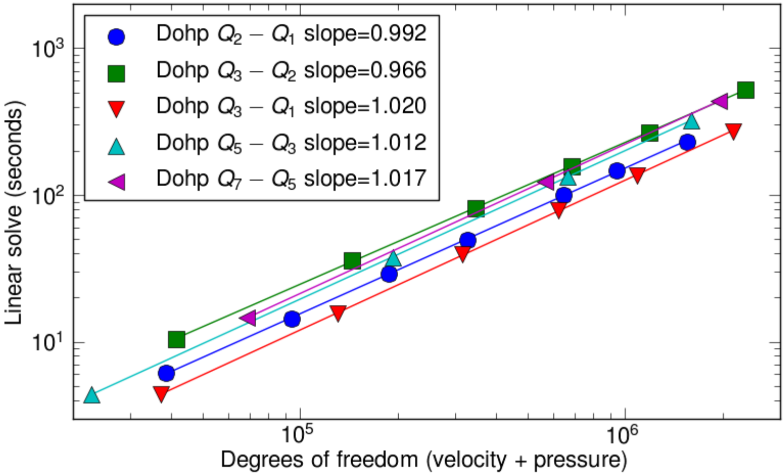
ALE form

After discretization in time ($\alpha \propto 1/\Delta t$) we have a Jacobian

$$\begin{bmatrix} A_{II} & A_{I\Gamma} & & & & \\ & \alpha M_{\Gamma\Gamma} & & -N_{\Gamma\Gamma} & & \\ G_{II} & G_{\Gamma I} & B_{II} & B_{I\Gamma} & C_I^T & D_I \\ G_{I\Gamma} & G_{\Gamma\Gamma} & B_{\Gamma I} & B_{\Gamma\Gamma} & C_\Gamma^T & D_\Gamma \\ G_{Ip} & G_{\Gamma p} & C_I & C_\Gamma & & \\ \alpha E_I & \alpha E_\Gamma & F_I & F_\Gamma & & \alpha M_\Theta + J \end{bmatrix} \begin{bmatrix} x_I \\ x_\Gamma \\ u_I \\ u_\Gamma \\ p \\ \Theta \end{bmatrix}$$

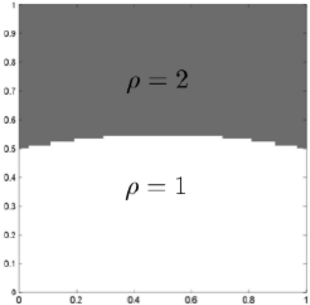
- pseudo-elasticity for mesh motion
- $(\dot{x} - u) \cdot n =$ accumulution
- “just” geometry
- Stokes problem
- temperature dependence of rheology
- convective terms and strain heating in heat transport
- thermal advection-diffusion

Power-law Stokes Scaling



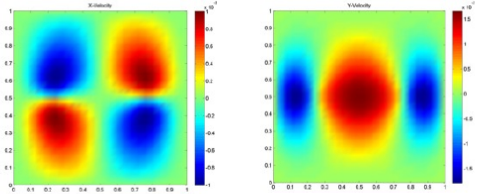
Only assembles Q_1 matrices, ML for elliptic pieces

Artifacts of stabilization

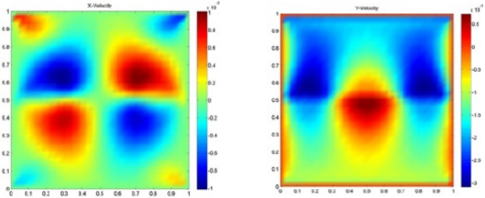


Rayleigh-Taylor initiation, isoviscous
(Dave May and Yury Mishin)

$Q_2 - P_{-1}$ (stable, locally conservative)



$Q_1 - Q_1$ (stabilized)



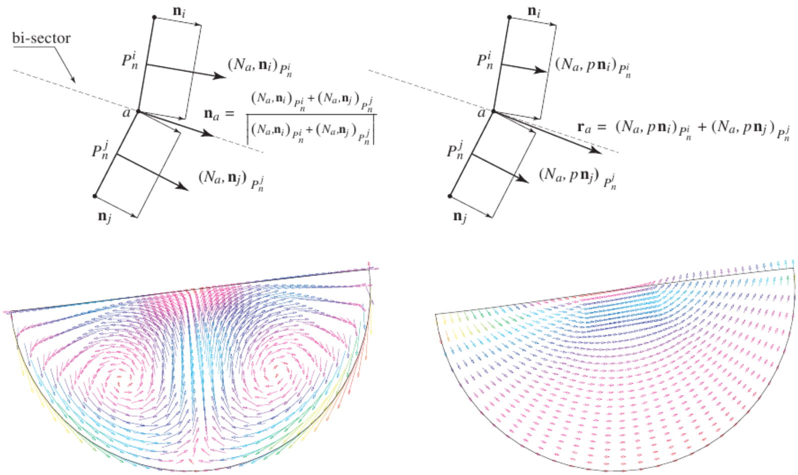
u v

Construction of conservative nodal normals

$$n^i = \int_\Gamma \phi^i n$$

- Exact conservation even with rough surfaces
- Definition is robust in 2D and for first-order elements in 3D
- $\int_\Gamma \phi^i = 0$ for corner basis function of undeformed P_2 triangle
- May be negative for sufficiently deformed quadrilaterals
- Mesh motion should use normals from CAD model
  - Difference between CAD normal and conservative normal introduces correction term to conserve mass within the mesh
  - Anomolous velocities if disagreement is large (fast moving mesh, rough surface)
- Normal field not as smooth/accurate as desirable (and achievable with non-conservative normals)
  - Mostly problematic for surface tension
  - Walkley et al, *On calculation of normals in free-surface flow problems*, 2004

Need for well-balancing



(Behr, *On the application of slip boundary condition on curved surfaces*, 2004)

“No” boundary condition

- Integration by parts produces

$$\int_\Gamma v \cdot T\boldsymbol{\sigma} \cdot n, \qquad \boldsymbol{\sigma} = \eta Du - p1, \qquad T = 1 - n \otimes n$$

- Continuous weak form requires either
  - Dirichlet: $u|_\Gamma = f \implies v|_\Gamma = 0$
  - Neumann/Robin: $\boldsymbol{\sigma} \cdot n|_\Gamma = g(u,p)$
- Discrete problem allows integration of $\boldsymbol{\sigma} \cdot n$ “as is”
  - Extends validity of equations to include Γ
  - *Not* valid for continuum equations
  - Introduced by Papanastasiou et al, 1992 for outflow boundaries
  - Griffiths 1997, Renardy 1997, Behr 2004

Outlook

- Exact local conservation is critical for problems with discontinuous geometry and coefficients
- Nonlinear slip on irregular surfaces is hard but tractable (mostly)
- Smooth manufactured solutions are necessary, but not sufficient to study solver and discretization performance
- Need good software to combine relaxation for loosely coupled processes and factorization for stiff/indefinite coupling
- Modeling of boundary layer processes in highly anisotropic geometry likely requires conforming to the interface

Tools

- PETSc `http://mcs.anl.gov/petsc`
  - ML, Hypre, MUMPS
- ITAPS `http://itaps.org`
  - MOAB, CGM, Lasso