Scalable Full-Stokes Ice Sheet Simulation

Toby Isaac[1], Carsten Burstedde[1], Georg Stadler[1], Omar Ghattas[123]

NCAR CESM Workshop, Breckenridge, June 23, 2011

[1] Institute for Computational Engineering and Sciences, UT-Austin
[2] Jackson School of Geosciences, UT-Austin
[3] Dept. of Mechanical Engineering, UT-Austin

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Our approach to ice sheet dynamics

- Solve for 3D flow using incompressible Stokes with nonlinear viscosity governed by Glen's flow law
- Adaptive mesh refinement to efficiently allocate computation where detail is needed
- Finite element discretization of PDEs
- Linear solvers that are highly scalable to both large problem sizes and a large number of processors

Full Stokes Equations

Equations of motion

$$-\nabla\cdot[\mu(T,\boldsymbol{u})\,\dot{\varepsilon}-\boldsymbol{I}p]=\rho\boldsymbol{g}, \qquad [\dot{\varepsilon}=\tfrac{1}{2}(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)]$$

$$\nabla\cdot\boldsymbol{u}=0,$$

$$\rho c\left(\frac{\partial T}{\partial t}+\boldsymbol{u}\cdot\nabla T\right)-\nabla\cdot(K\nabla T)=2\mu\,\mathrm{tr}(\dot{\varepsilon}^2)$$

Constitutive relations

$$\mu(T,\boldsymbol{u})=\left\{A_0\exp\left(-\frac{Q}{RT}\right)\right\}^{-\frac{1}{n}}\dot{\varepsilon}_{\mathrm{II}}^{\frac{1-n}{2n}} \qquad [\dot{\varepsilon}_{\mathrm{II}}=\tfrac{1}{2}\mathrm{tr}(\dot{\varepsilon}^2)]$$

Boundary conditions

$$T|_{\Gamma_{FS}}=T_{FS}, \quad \frac{Dz}{Dt}|_{\Gamma_{FS}}=a, \qquad \sigma\boldsymbol{n}|_{\Gamma_{FS}}=\boldsymbol{0},$$

$$K\nabla T\cdot\boldsymbol{n}|_{\Gamma_B}=q_B, \quad \boldsymbol{u}\cdot\boldsymbol{n}|_{\Gamma_B}=0, \quad (\boldsymbol{I}-\boldsymbol{n}\otimes\boldsymbol{n})(\sigma\boldsymbol{n}+\beta\boldsymbol{u})|_{\Gamma_B}=\boldsymbol{0}$$

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Constitutive relations

Arrhenius thinning near melting point

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Constitutive relations

shear thinning with second invariant

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Full Stokes Equations

Equations of motion

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$$\mu(T, \boldsymbol{u}) = \left\{ A_0 \exp\left(-\frac{Q}{RT} \right) \right\}^{-\frac{1}{n}} \dot{\varepsilon}_{\mathrm{II}}^{\frac{1-n}{2n}} \qquad [\dot{\varepsilon}_{\mathrm{II}} = \tfrac{1}{2}\mathrm{tr}(\dot{\varepsilon}^2)]$$

Boundary conditions

dynamic free surface

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Boundary conditions basal friction

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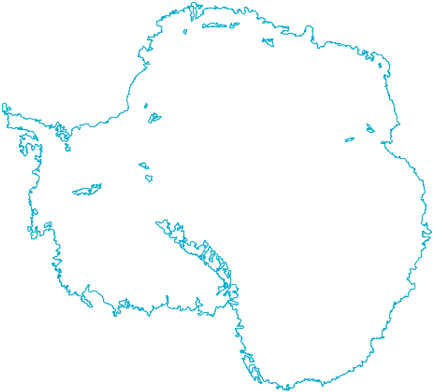
Coarse mesh generation

Thanks to Andrew Sheng



Obtain map view boundary from ice thickness data

Coarse mesh generation

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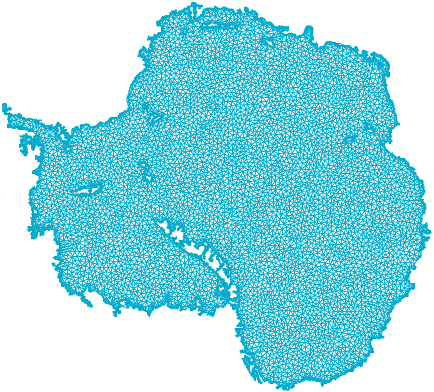

Triangulate with Shewchuk's *triangle*

- maximum area constrained by desired element aspect ratio

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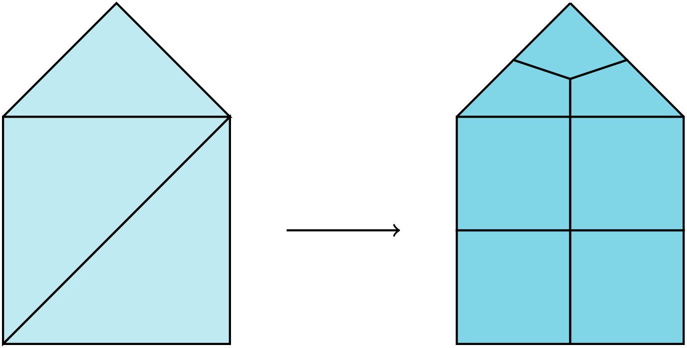

Combine two adjacent triangles into four quads when possible, split remaining triangles into three quads

Coarse mesh generation

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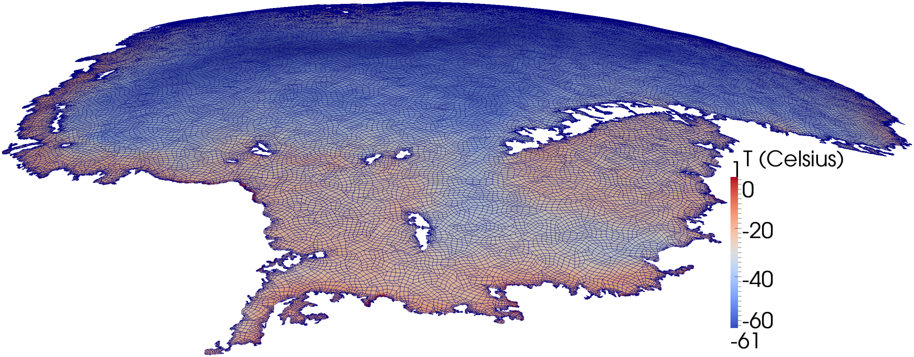
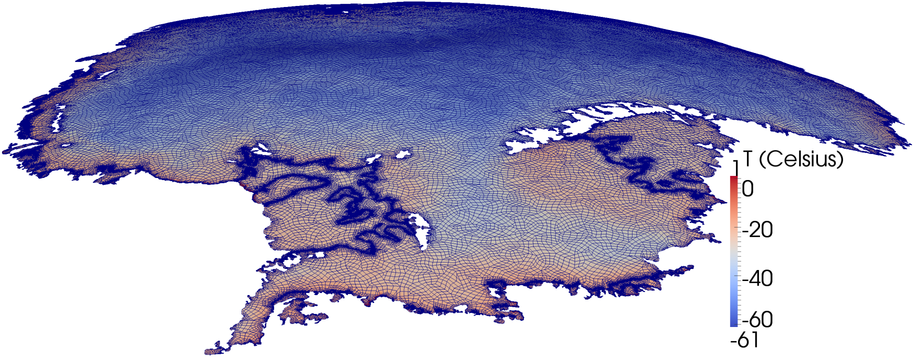
Extrude to 3D: ∼47,000 octrees, mesh resolution ∼25 km, or less as dictated by geometry

Mesh refinement



Additional resolution needed at transition from grounded to floating ice: ∼2 km

Mesh refinement



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Scalability of Stokes iterative solver on example problem

#cores	#elem/ core	#dofs	#iter	setup time [s]	matvec time [s]	Vcycle time [s]
8	57K	2.6M	378	11.8	0.52 (1)	0.82 (1)
16	28K	2.6M	370	6.4	0.27 (0.96)	0.63 (0.65)
32	14K	2.6M	360	3.0	0.14 (0.96)	0.33 (0.61)
64	7.1K	2.6M	348	2.2	0.071 (0.92)	0.17 (0.60)
64	48K	16M	264	14.1	0.40 (1)	1.09 (1)
128	24K	16M	251	12.3	0.21 (0.96)	0.60 (0.90)
256	12K	16M	237	9.4	0.11 (0.93)	0.31 (0.87)
512	6K	16M	244	10.2	0.055 (0.91)	0.17 (0.79)
512	47K	111M	242	28.4	0.39 (1)	1.13 (1)
1024	24K	111M	237	18.8	0.21 (0.96)	0.58 (0.98)
2048	12K	111M	232	34.7	0.10 (0.97)	0.32 (0.88)
4096	6K	111M	214	67.0	0.050 (0.97)	0.21 (0.69)

Test environment

- Strong scaling for three different problem sizes on TACC's Ranger
- Number of MINRES iterations to decrease preconditioned residual by factor of 10^3
- AMG setup and V-cycle time based on *ML* from *Trilinos* with RCB/Zoltan repartitioning within multigrid hierarchy

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Test parameters

- Trilinear elements
- Temperature dependent viscosity varies over an order of magnitude
- Basal boundary condition: no flow above a certain elevation, no tangential stress below (free tangential slip)

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Notice lower iteration counts with larger problem sizes: adding layers reduces boundary effects

Numerical advances

$$A_{\text{old}} = \begin{pmatrix} V_1 & \nabla \\ \nabla\cdot & -C_{\text{stab}} \end{pmatrix} \qquad A_{\text{new}} = \begin{pmatrix} V_q & \nabla \\ \nabla\cdot & 0 \end{pmatrix}$$

$$P_{\text{old}} = \begin{pmatrix} V_{1,\text{AMG}} & 0 \\ 0 & S_{\text{diag}} \end{pmatrix} \qquad P_{\text{new}} = \begin{pmatrix} \tilde{V}_{q,\text{AMG}} & \nabla \\ 0 & S_{BFB^T} \end{pmatrix}$$

Old	New
Linear approximation of velocity	Higher order (e.g. 2, 3,4) approximation of velocity
Compressible linear approximation of pressure	Stable, locally incompressible pressure approximation
Diagonal Schur preconditioning that fails to precondition β	BFB^T Schur preconditioning that can handle highly variable viscosity and β
Block diagonal preconditioner	Block upper triangular preconditioner

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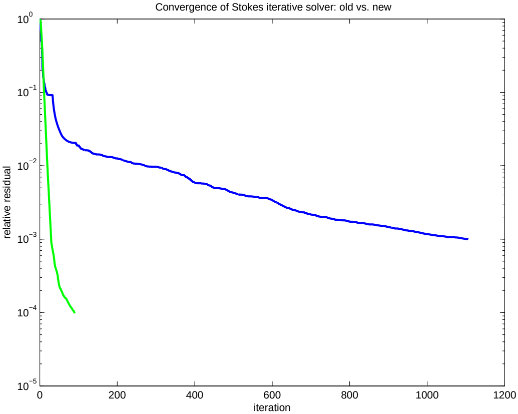
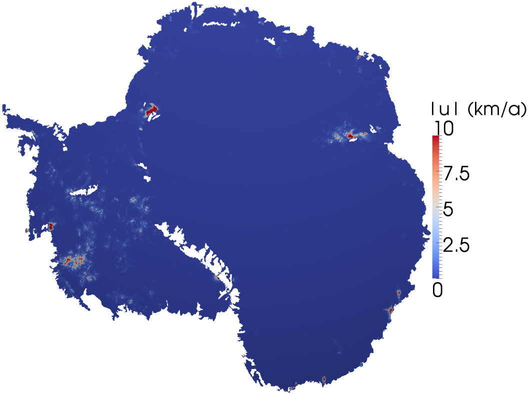
Numerical method comparison



- Identical β conditions and rhs
- Convergence norms are slightly different, but equivalent
- Work per iteration for new method roughly 4 times that of old method

Topography spoils convergence

Example solution of full nonlinear equations



- Non-slip basal conditions
- relative tolerance $\varepsilon = 10^{-3}$ for nonlinear solver
- Nonlinear iteration stopped based on $< 1\%$ change in solution

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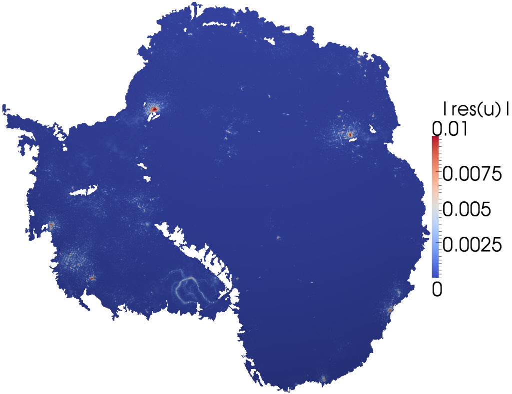

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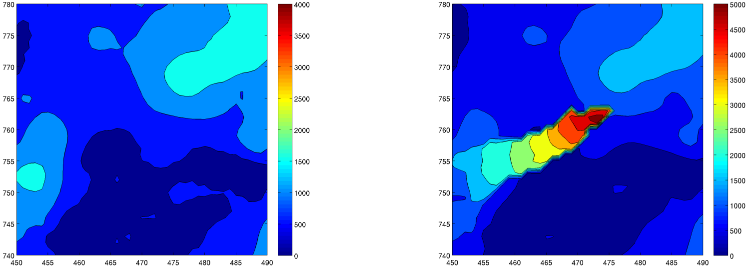
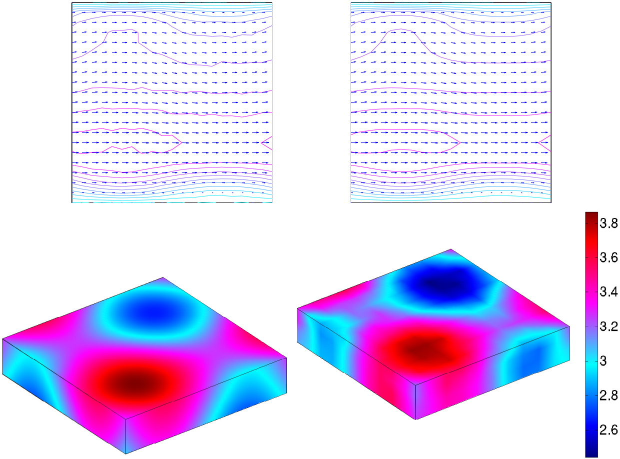


- SeaRISE Antarctica dataset with shelves: floating ice made thicker to satisfy flotation condition
- In the process of modifying Le Brocq's dataset to substitute
  - How should meteoric-only ice models handle geometry that includes firn?
- Incorporating new East Antarctica flight line data (Don Blankenship, Duncan Young, UTIG)

Concurrent work

Now that the linear solver can handle variable β, once geometry irregularities are resolved we can begin work on continental-scale inversion of β from observed velocities.



Acknowledgements

- NSF OPP-0941678
- DOE ASCR DE-SC0002710
- Computing time on TACC's Lonestar 4 system