Implementation and comparison of finite element methods for higher-order ice sheet models

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in collaboration with FSU, LANL, Oak Ridge, Sandia

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Ice Sheet Modeling

Main components of an ice model:

Ice flow equations
(momentum and mass balance).

Model for the evolution of the boundaries.
(thickness evolution equation)

Temperature equation.

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LifeV
(finite elements)

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MPAS
(finite volumes)

Temperature equation.

Ice Sheet Modeling

"Reference" model: FULL ***STOKES***[1]

Approximations based on their accuracy with respect to the ice-sheet aspect ratio δ

$O(\delta)$ ***FO,*** Blatter-Pattyn first order model[2] (3D PDE, in horizontal velocities)

$O(1)$ Zeroth order, depth integrated models:
SIA, Shallow Ice Approximation (slow sliding regimes) ,
SSA Shallow Shelf Approximation (2D PDE) (fast sliding regimes)

$\simeq O(\delta)$ High order, depth integrated (2D) models: ***L1L2***[3], (L1L1)...

[1] *Gagliardini and Zwinger, 2008. The Cryosphere.*

[2] *Dukowicz, Price and Lipscomb, 2010. J. Glaciol.*

[3] *Schoof and Hindmarsh, 2010. Q. J. Mech. Appl. Math.*

Implementation Overview

Trilinos:
- Parallel Data Structures (EPETRA)
- Parallel Linear Solvers (GMRES, CG...)
- Preconditioners (Multilevel, Multigrid, Incomplete LU)
- Nonlinear Solvers (**NOX** package: Newton, JFNK methods)

⇧

LifeV
Parallel, object oriented, C++ Finite Element Library:
- linear and quadratic finite elements
- assembling of finite element matrices
- handling of boundary conditions

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MPAS (land ice model):
- Voronoi unstructured grids
- Evolution equation solvers (temperature and thickness equation)
- ...

Why Finite Elements?

Finite Element features:

• Particularly suited for s.p.d. elliptic PDEs, as the ones occurring in FO, L1L2, SSA models

• Complex geometries (unstructured, nonuniform grids can be used)

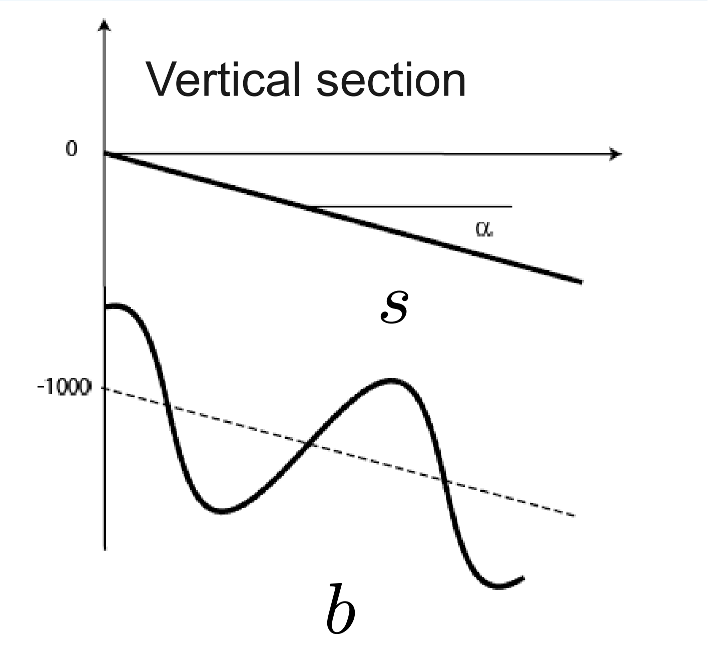
• Several boundary conditions (free slip, free stress, sliding b.c., coulomb friction) are naturally handled (no need to use ghost cells)

• Accuracy: good description of the boundary and possibility of using high order elements.

ISIMP-HOM Benchmarks: Test C

The geometry is a parallelepiped with square base.
Horizontal length L.

$$s(x,y) = -x\ \tan(\alpha) \qquad \textit{elevation}$$
$$b(x,y) = s(x,y) - 1 \qquad \textit{bedrock}$$



Boundary conditions:
stress free at the elevation
sliding at the bedrock

$$-\mu\frac{\partial u}{\partial n} = u + u\sin\left(\tfrac{2\pi}{L}x\right)\sin\left(\tfrac{2\pi}{L}y\right)$$

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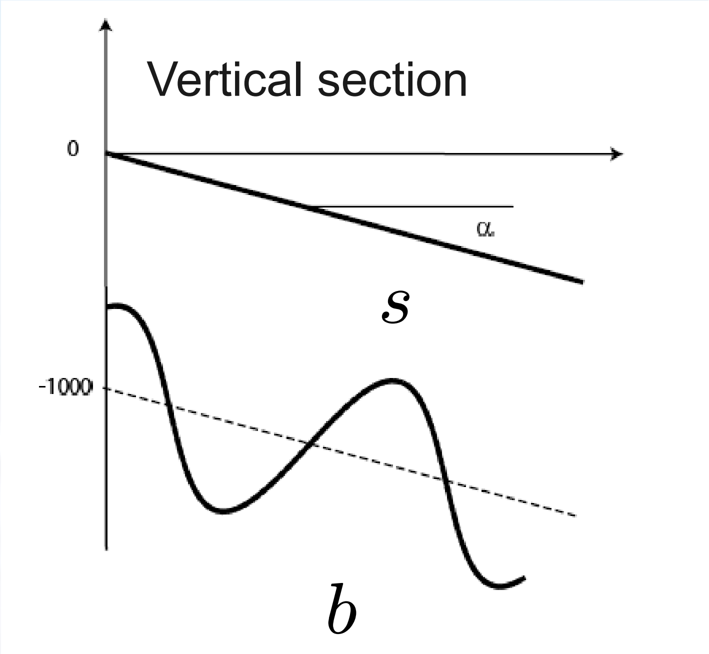

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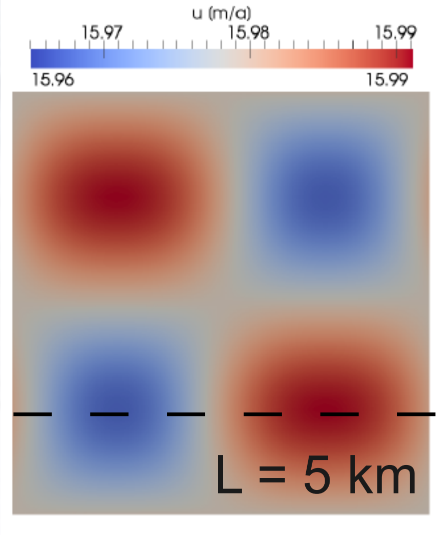
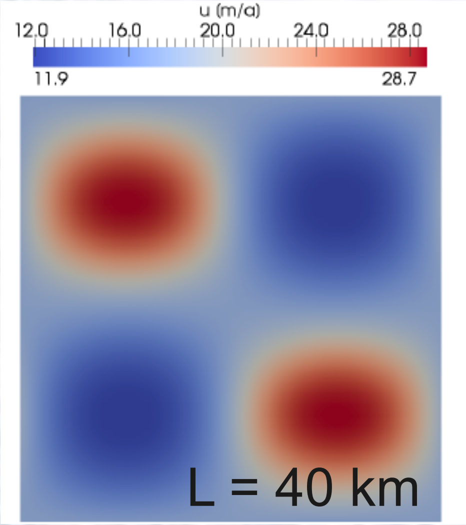
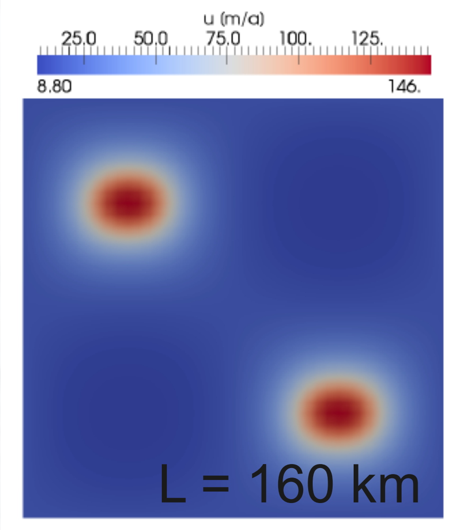
surface velocity

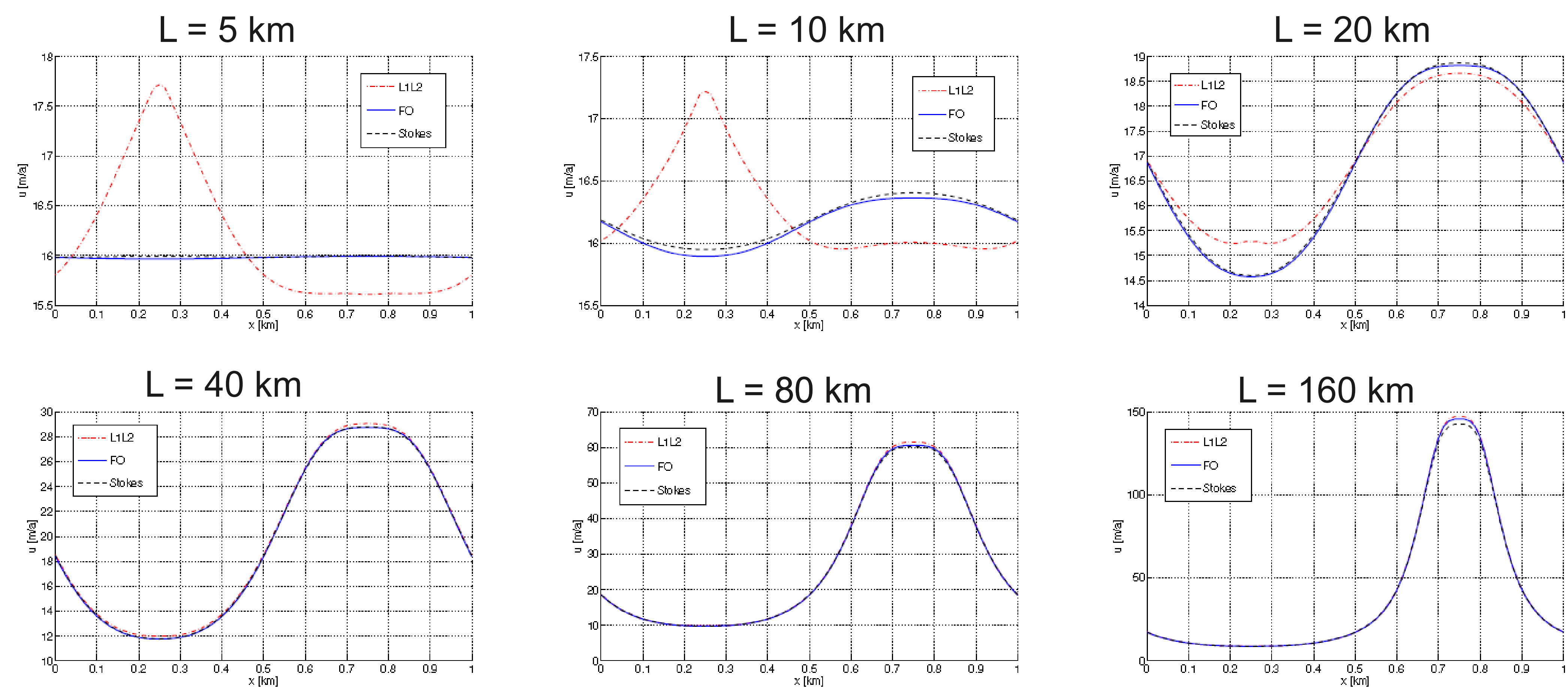






ISIMP-HOM: Test C

Surface velocity



Test C: convergence of the nonlinear method

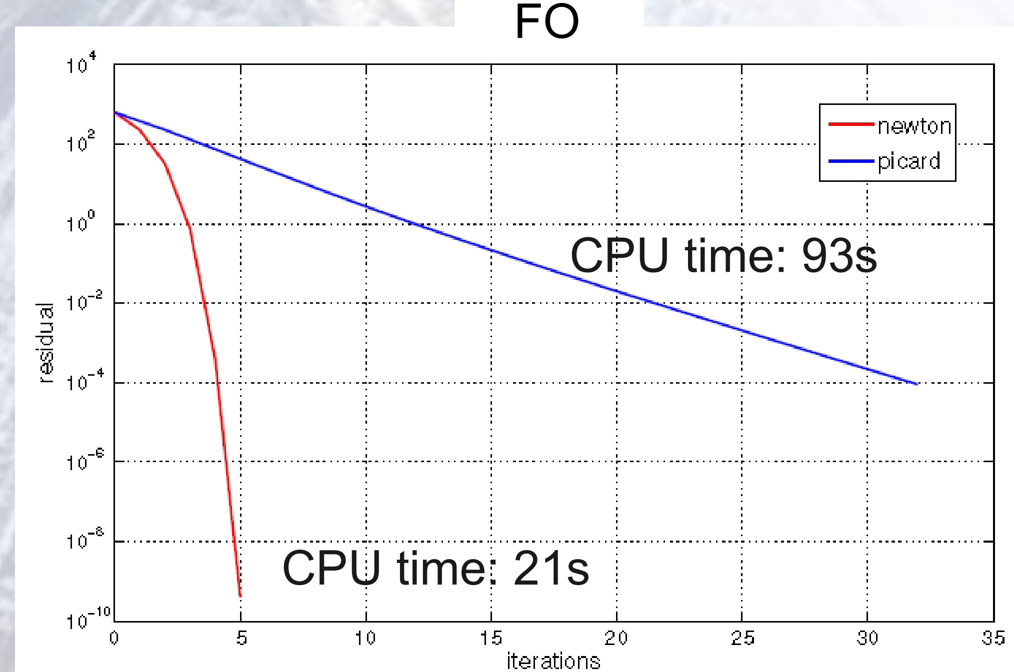
Solve $\quad F(u) = 0.$

Newton method: $\quad J^k\,(\delta u)^{k+1} = -F(u^k).$ (NOX library)

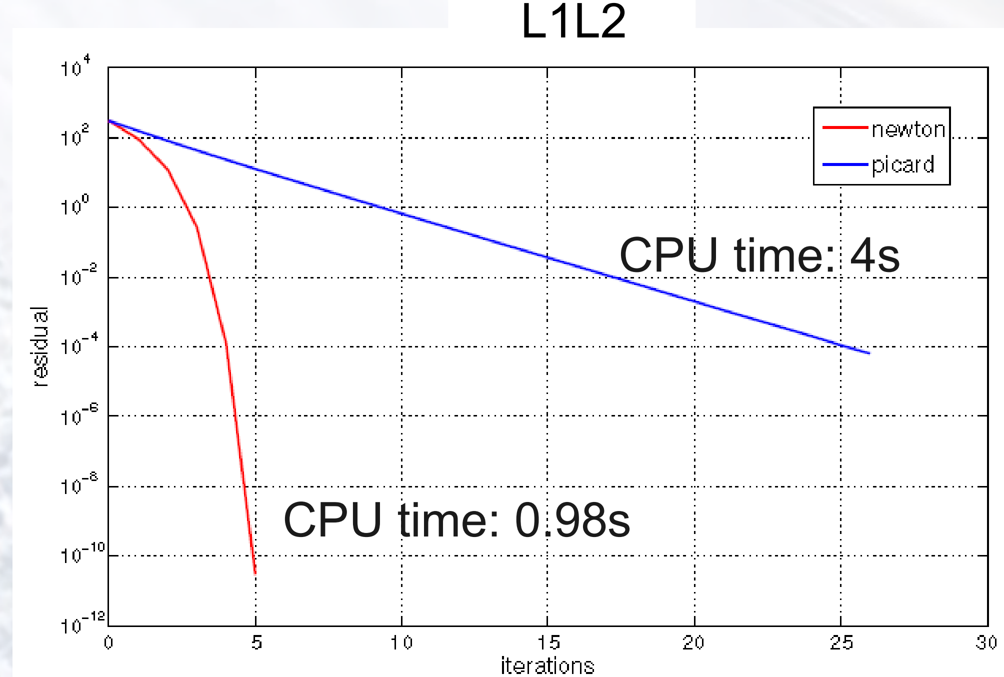
The exact Jacobian matrix must be assembled at each nonlinear iteration.

In order to increase robustness of Newton method, the Newton step is halved to achieve monotonic convergence.

FO

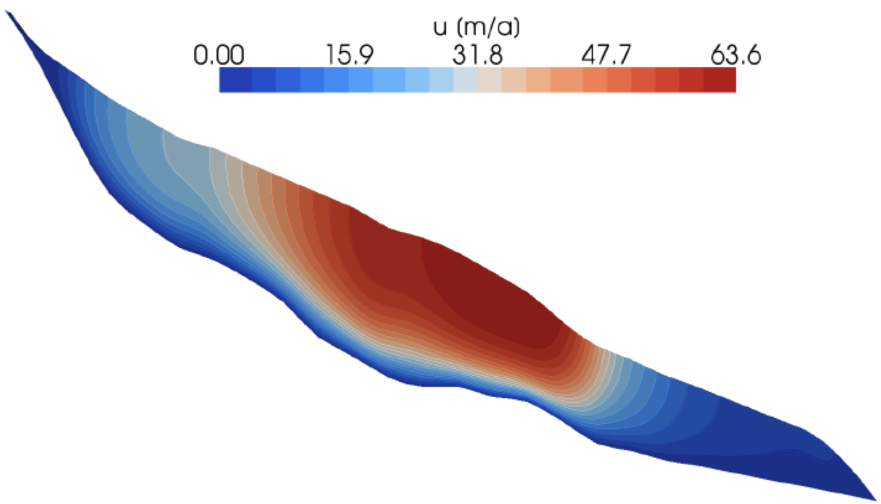


L1L2

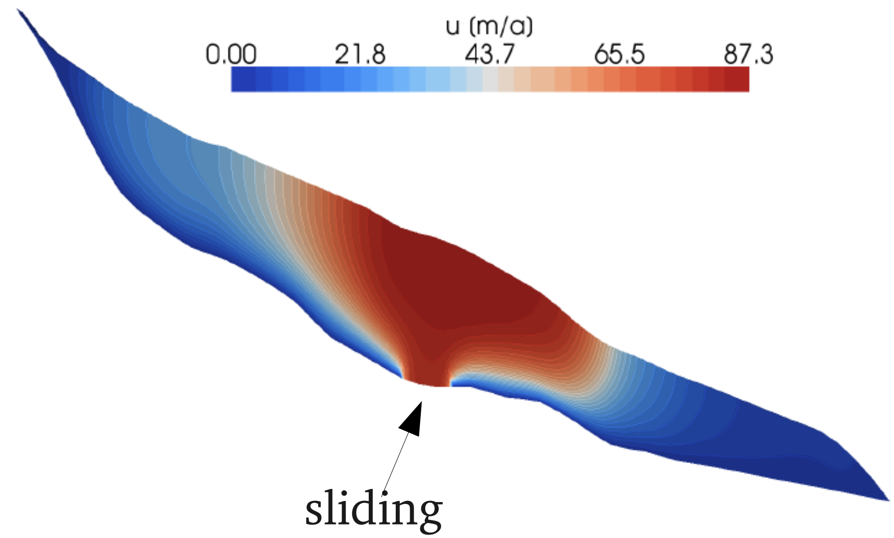


Test E: Glacier d'Arolla

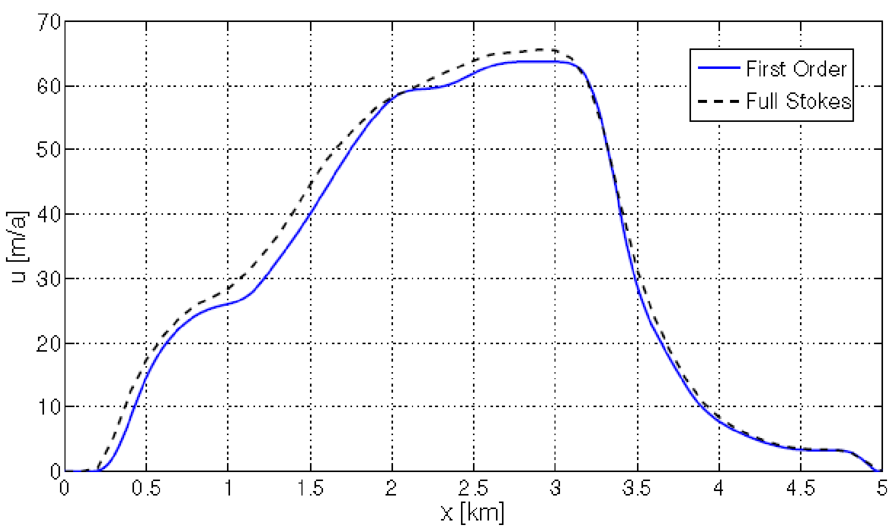
No Slip

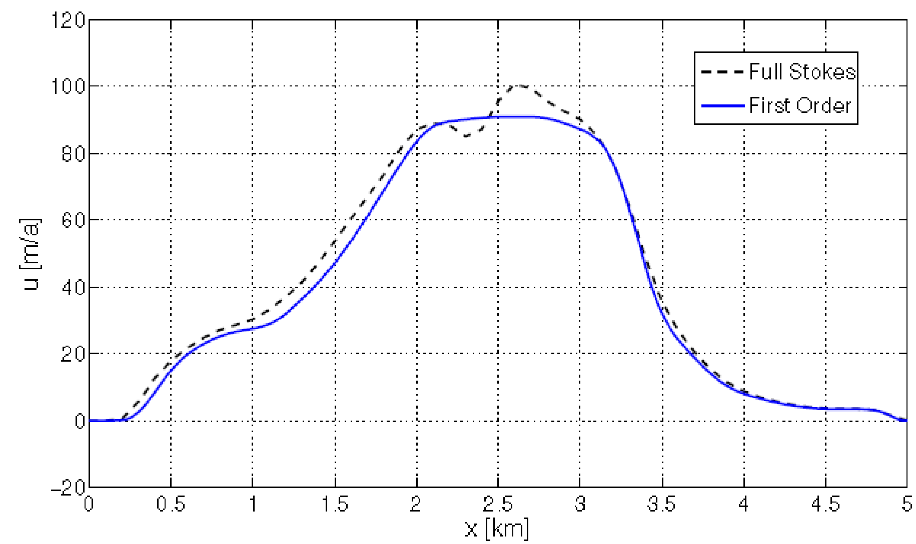


Partial sliding

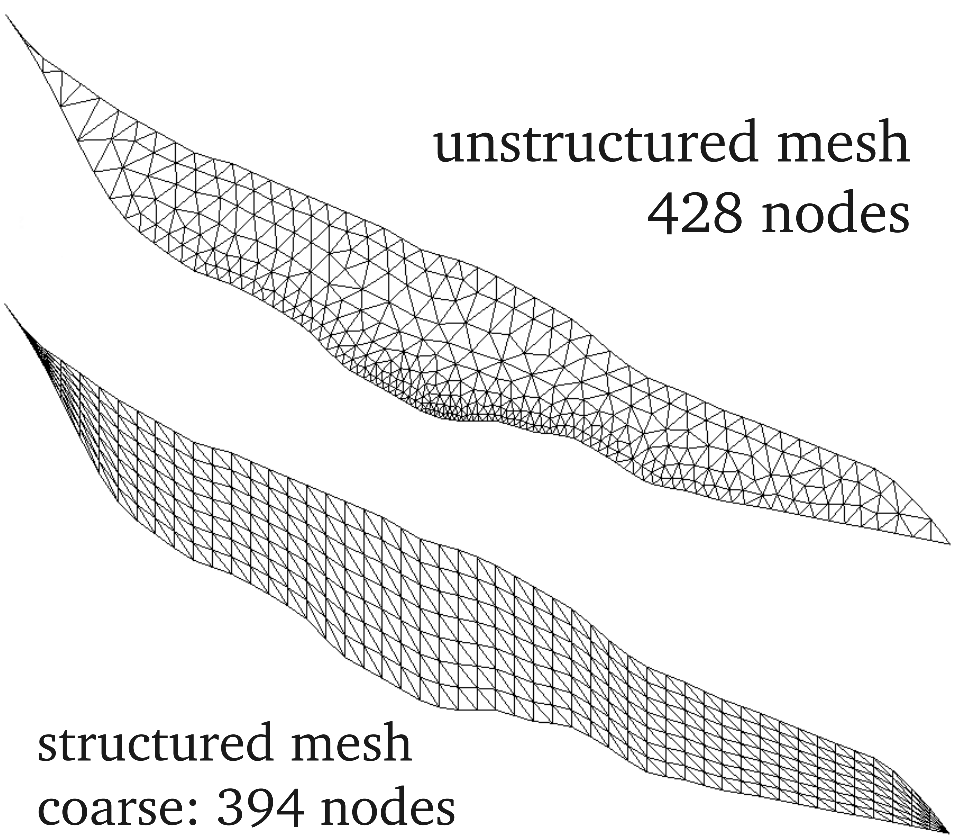


Surface velocity



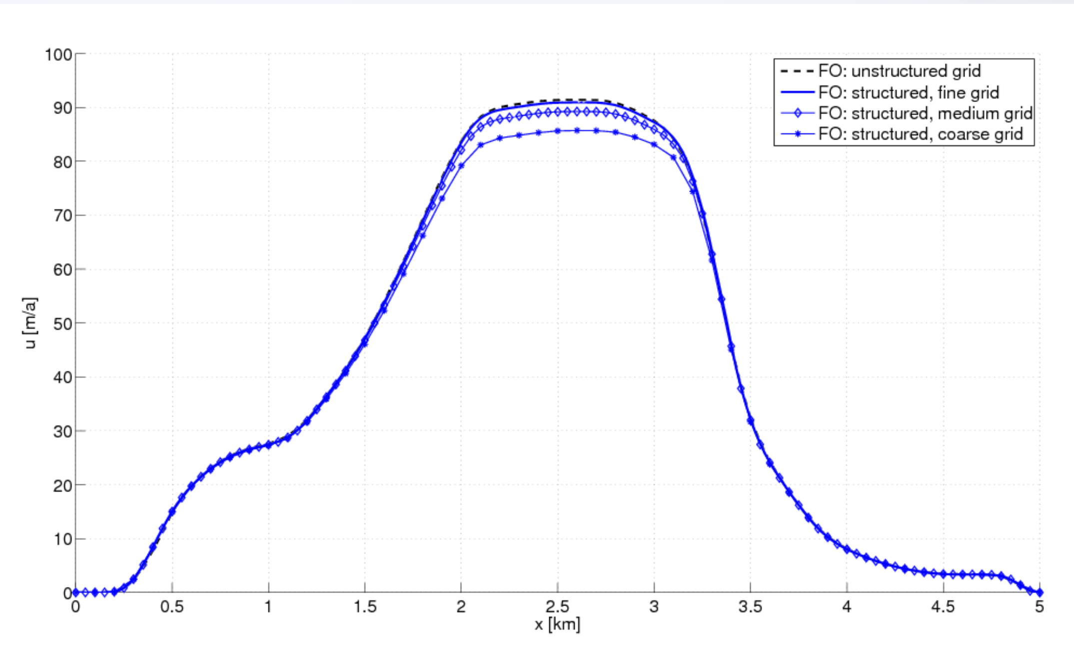


Test E: Glacier d'Arolla
comparison between structured and unstructured adaptive grids



unstructured mesh
428 nodes

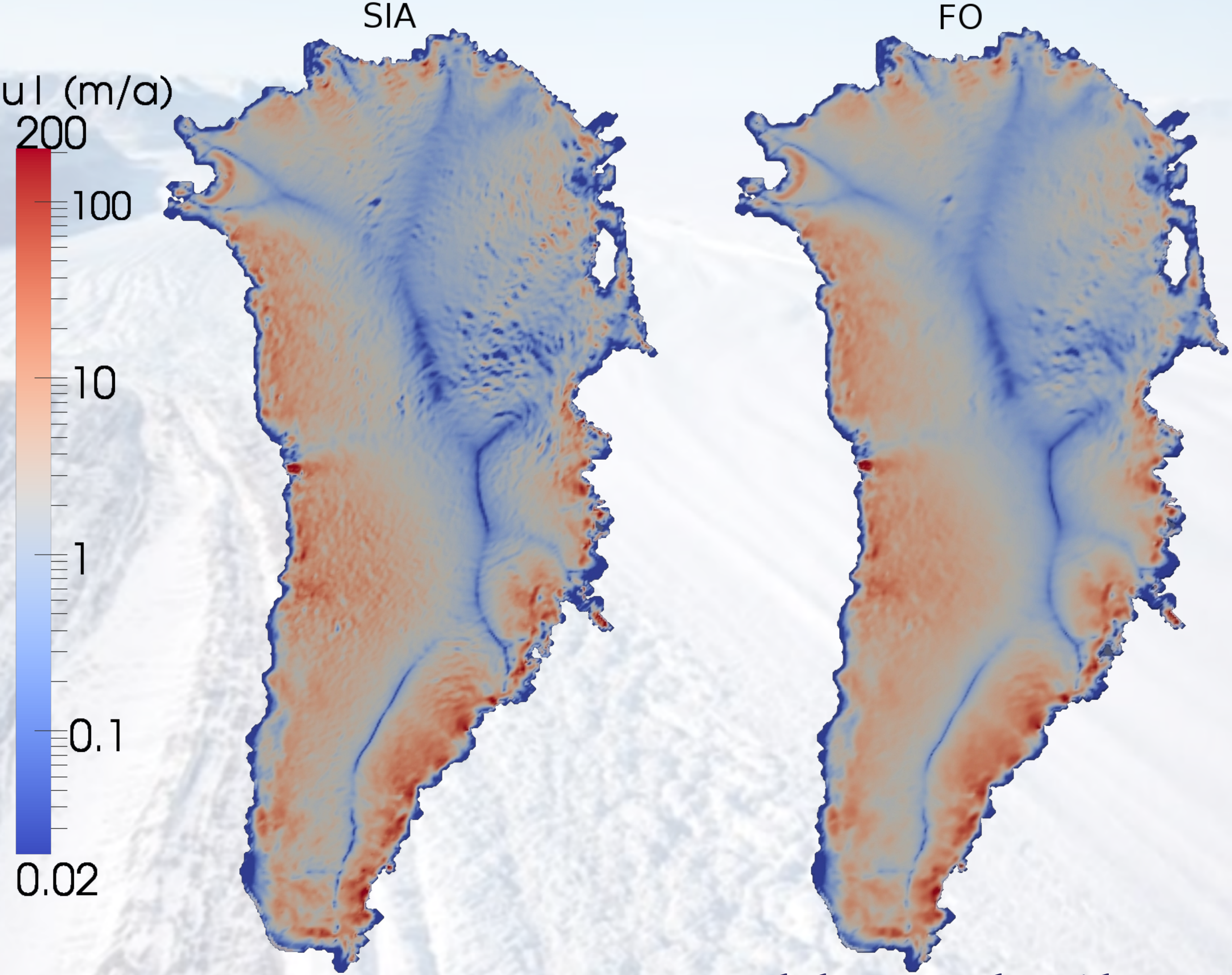
structured mesh
coarse: 394 nodes
fine: 2978 nodes



To obtain the same accuracy of the unstructured mesh, we need a structured mesh which is seven times bigger.

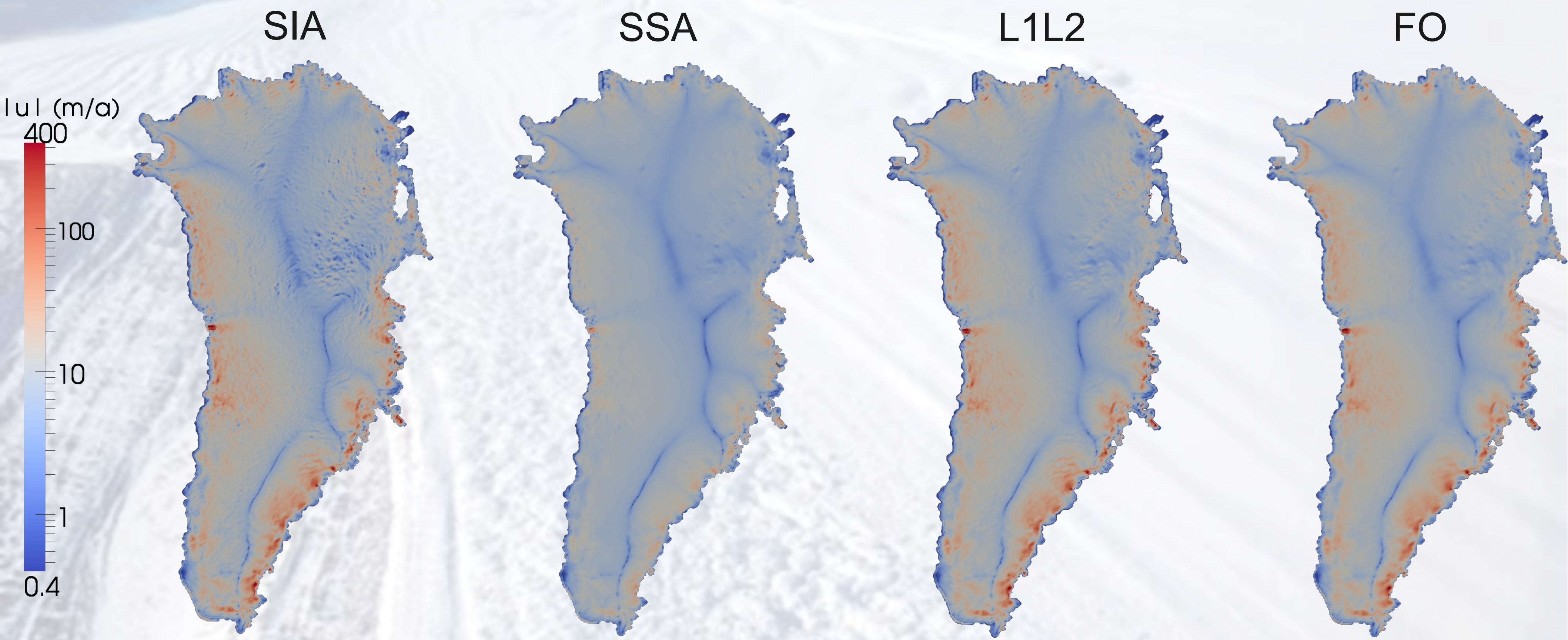
Towards realistic simulations:

Greenland, no-slip case



Towards realistic simulations:
Greenland, sliding case

Comparisons between different models



Future development

- Interfacing with MPAS

- Code Optimization, testing and improving scalability

- Comparisons on real geometries
  - FO vs L1L2
  - Linear vs Quadratic elements
  - Structured vs unstructured, adaptive meshes

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K. Evans, J. Nichols (Oak Ridge)

A. Salinger (Sandia)

Ice Sheet Modeling

Velocity equations (momentum and mass balance)

$$-\nabla \cdot \sigma = \rho \mathbf{g} \qquad \text{and} \qquad \nabla \cdot \mathbf{u} = 0,$$

with $\sigma = \tau - pI = 2\mu\dot{\varepsilon} - pI$,
where μ viscosity, $\dot{\varepsilon}$ shear rate

LifeV
(finite elements)

Thickness (H) evolution equation

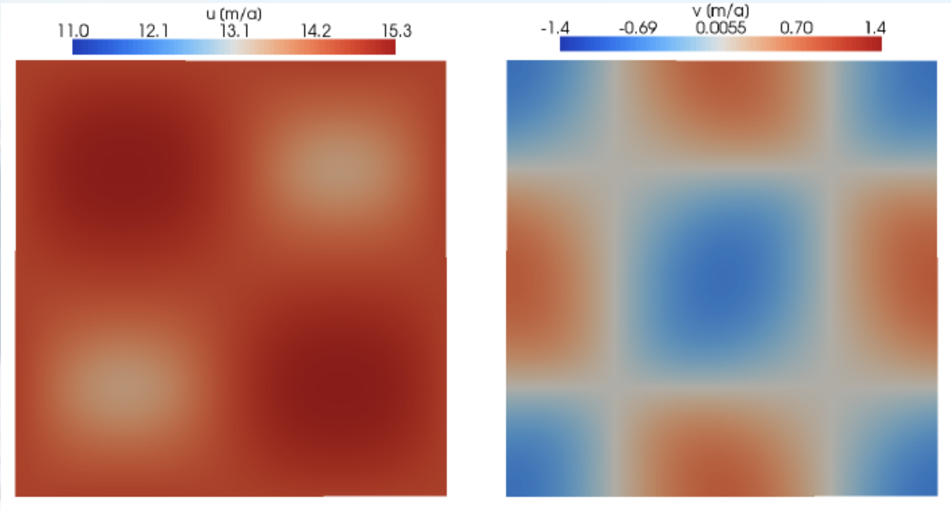
$$\frac{\partial H}{\partial t} = H_{flux} - \nabla \cdot \int_z \mathbf{u}\, dz$$

Temperature equation

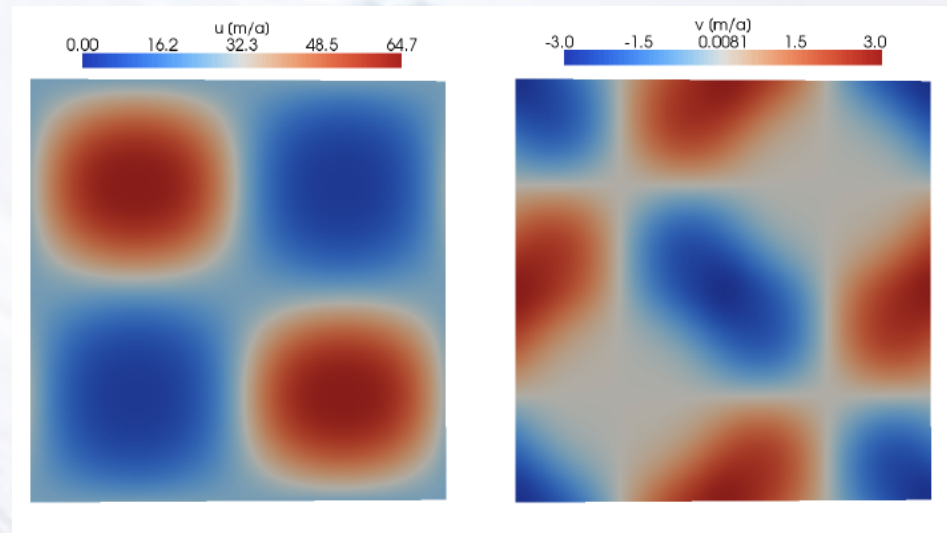
$$\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial z}\left(k \frac{\partial T}{\partial z}\right) - \rho c \mathbf{u} \cdot \nabla T + 2\dot{\varepsilon}\sigma$$

ISIP-HOM Test A

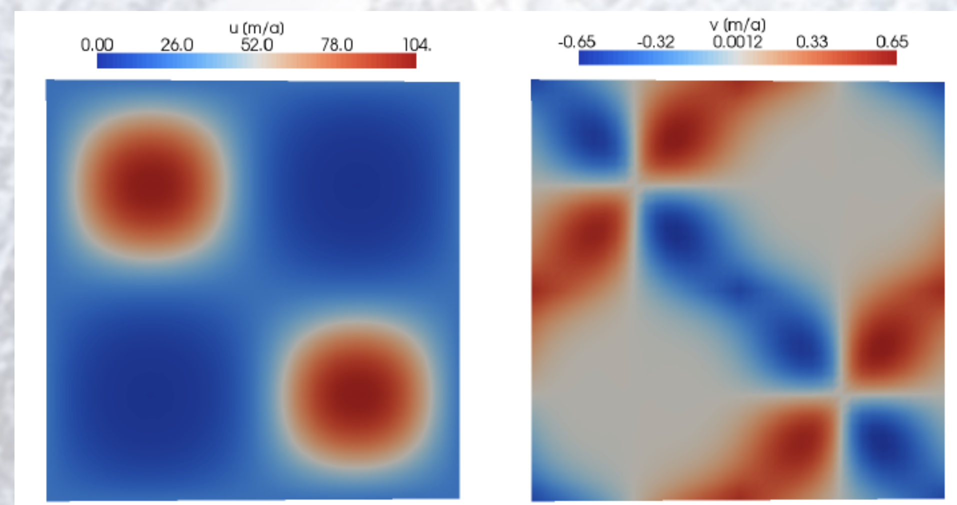
surface velocity



L = 5 km



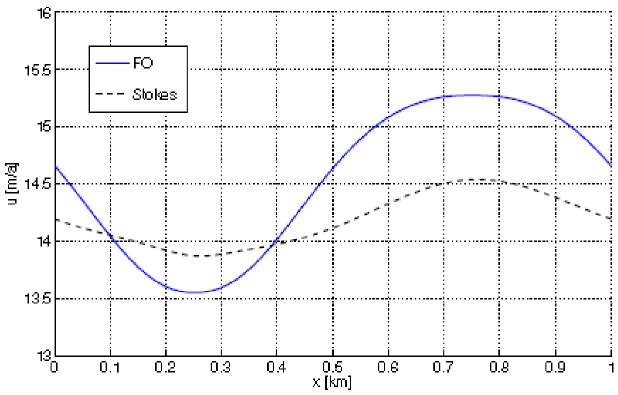
L = 40 km



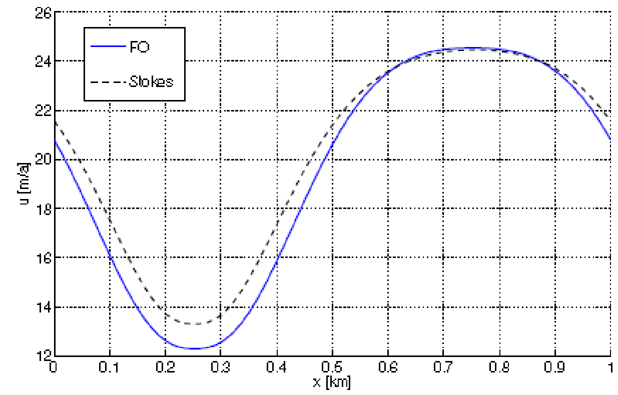
L = 160 km

ISIP-HOM Test A
surface velocity

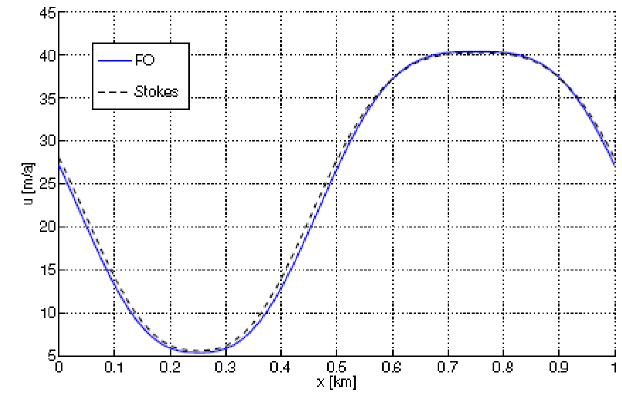
L = 5 km



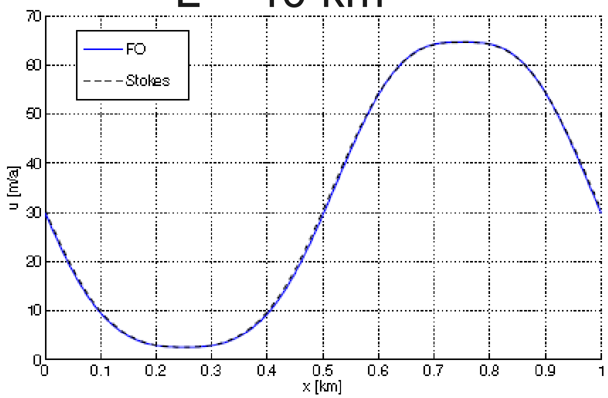
L = 10 km



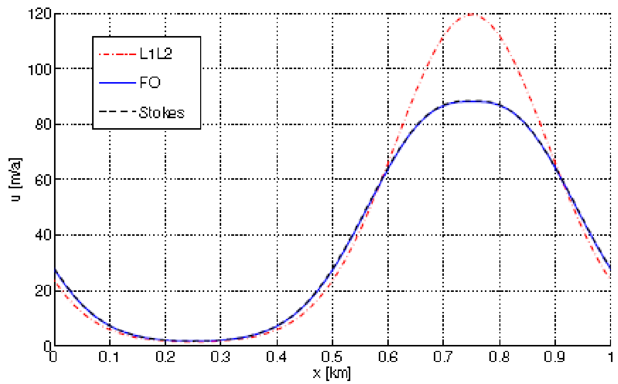
L = 20 km



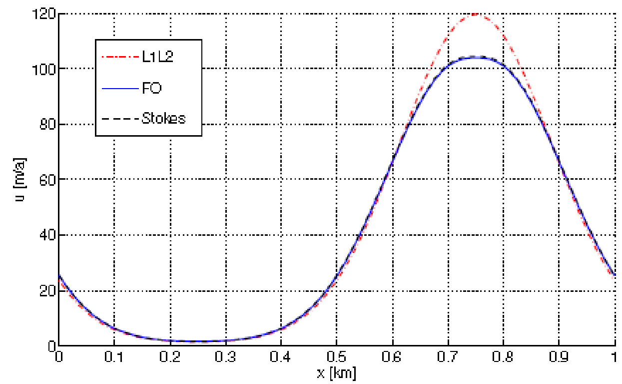
L = 40 km



L = 80 km



L = 160 km



First Order Equations

$$
\begin{cases}
\frac{\partial}{\partial x}\left\{2\mu\left(2\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right\}+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}+\frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right) & = & \rho g\frac{\partial s}{\partial x}\\
\frac{\partial}{\partial x}\left\{\mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right\}+\frac{\partial}{\partial y}\left\{2\mu\left(\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}\right)\right\}+\frac{\partial}{\partial z}\left(\mu\frac{\partial v}{\partial z}\right) & = & \rho g\frac{\partial s}{\partial y},
\end{cases}
$$

where s is the ice surface.

$$\mu = \frac{1}{2}A^{-\frac{1}{n}}\dot{\varepsilon}_e^{\left(\frac{1}{n}-1\right)} \qquad \dot{\varepsilon}_e = \sqrt{\dot{\varepsilon}_{xx}^2+\dot{\varepsilon}_{yy}^2+\dot{\varepsilon}_{xx}\dot{\varepsilon}_{yy}+\dot{\varepsilon}_{xy}^2+\dot{\varepsilon}_{xz}^2+\dot{\varepsilon}_{yz}^2}$$

FO is a nonlinear system of elliptic equations in the horizontal velocities.

Typical boundary conditions:
- free stress at the surface,
- no-slip or sliding boundary conditions at the interface.

First Order Equations

$$\begin{cases} \dfrac{\partial}{\partial x}\left\{2\mu\left(2\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}\right)\right\}+\dfrac{\partial}{\partial y}\left\{\mu\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right)\right\}+\dfrac{\partial}{\partial z}\left(\mu\dfrac{\partial u}{\partial z}\right) & = & \rho g\dfrac{\partial s}{\partial x} \\ \dfrac{\partial}{\partial x}\left\{\mu\left(\dfrac{\partial u}{\partial y}+\dfrac{\partial v}{\partial x}\right)\right\}+\dfrac{\partial}{\partial y}\left\{2\mu\left(\dfrac{\partial u}{\partial x}+2\dfrac{\partial v}{\partial y}\right)\right\}+\dfrac{\partial}{\partial z}\left(\mu\dfrac{\partial v}{\partial z}\right) & = & \rho g\dfrac{\partial s}{\partial y}, \end{cases}$$

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Weak formulation

FO weak formulation

$$\begin{cases} \int_{\Omega} 2\mu\, \dot{\varepsilon}_1(u,\, v) \cdot \nabla \varphi_1 d\mathbf{x} + \int_{\Gamma_\beta} \beta u \varphi_1 d\mathbf{s} + \int_{\Omega} \rho g \frac{\partial s}{\partial x} \varphi_1 d\mathbf{x} = 0 \\ \int_{\Omega} 2\mu\, \dot{\varepsilon}_2(u,\, v) \cdot \nabla \varphi_2 d\mathbf{x} + \int_{\Gamma_\beta} \beta v \varphi_2 d\mathbf{s} + \int_{\Omega} \rho g \frac{\partial s}{\partial y} \varphi_2 d\mathbf{x} = 0 \end{cases}$$

Nonlinear “stiff”

robin boundary condition for the sliding term

L1L2 weak formulation

$$\begin{cases} \int_{\Sigma} \left(2\bar{\mu}\, \dot{\varepsilon}_{1,b}(u,\, v) \cdot \nabla_{||} \varphi_1 + \beta u \varphi_1\right) d\Sigma = -\int_{\Sigma} \rho g H \frac{\partial s}{\partial x} \varphi_1 d\Sigma \\ \int_{\Sigma} \left(2\bar{\mu}\, \dot{\varepsilon}_{2,b}(u,\, v) \cdot \nabla_{||} \varphi_2 + \beta v \varphi_2\right) d\Sigma = -\int_{\Sigma} \rho g H \frac{\partial s}{\partial y} \varphi_2 d\Sigma \end{cases}$$

FO, Jacobian operator

$$J^k_{\mathcal{F}}(\delta_u, \delta_v) =$$

$$\begin{bmatrix} \displaystyle\int_\Omega 2(\mu')^k \left(\dot\varepsilon_1^k \cdot \nabla\delta_u + \dot\varepsilon_2^k \cdot \nabla\delta_v\right) \dot\varepsilon_1^k \cdot \nabla\varphi_1 d\mathbf{x} + \\ \qquad \displaystyle\int_\Omega 2\mu^k \dot\varepsilon_1(\delta_u, \delta_v) \cdot \nabla\varphi_1 d\mathbf{x} + \int_{\Gamma_\beta} \beta\delta_u\, \varphi_1 d\mathbf{s} \\ \displaystyle\int_\Omega 2(\mu')^k \left(\dot\varepsilon_1^k \cdot \nabla\delta_u + \dot\varepsilon_2^k \cdot \nabla\delta_v\right) \dot\varepsilon_2^k \cdot \nabla\varphi_2 d\mathbf{x} + \\ \qquad \displaystyle\int_\Omega 2\mu^k \dot\varepsilon_2(\delta_u, \delta_v) \cdot \nabla\varphi_2 d\mathbf{x} + \int_{\Gamma_\beta} \beta\delta_v\, \varphi_2 d\mathbf{s} \end{bmatrix},$$

$$(\mu')^k := \left.\frac{\partial\mu}{\partial\dot\varepsilon_e^2}\right|_{(\dot\varepsilon_e = \dot\varepsilon_e^k)} = \frac{1-n}{4n}\; A^{-\frac{1}{n}} (\dot\varepsilon_e^k)^{\left(\frac{1}{n}-3\right)} = \frac{1-n}{2n}\; \frac{\mu^k}{(\dot\varepsilon_e^k)^2}.$$

L1L2 model (Schoof Hindmarsh 2010)

Approximate stress and strain tensors:

$$\tau_e^2 \approx |\tau|_{\|}^2 + |\tilde{\tau}|_{\perp}^2$$

$$\dot{\varepsilon}_{ij}\big|_{z=b} = A\left(|\tau|_{\|}^2 + |\tilde{\tau}|_{\perp}^2\right)^{\frac{n-1}{2}} \tau_{ij} \quad \text{for } i, j \in \{x, y\}$$

Solve 2D equations:

$$\begin{cases} -\nabla \cdot (2\bar{\mu}\,\dot{\varepsilon}_{1,b}) + \beta u_b = -\rho g H \dfrac{\partial s}{\partial x} \\ -\nabla \cdot (2\bar{\mu}\,\dot{\varepsilon}_{2,b}) + \beta v_b = -\rho g H \dfrac{\partial s}{\partial y} \end{cases}$$

$$\bar{\mu}(x, y) = \int_b^s \mu(|\dot{\varepsilon}_b|_{\|}, |\tilde{\tau}|_{\perp}) dz$$

Recover velocity:

$$u = u_b + 2\int_b^z A\,\tau_e^{n-1}\tau_{xz} dz, \quad v = v_b + 2\int_b^z A\,\tau_e^{n-1}\tau_{yz} dz.$$

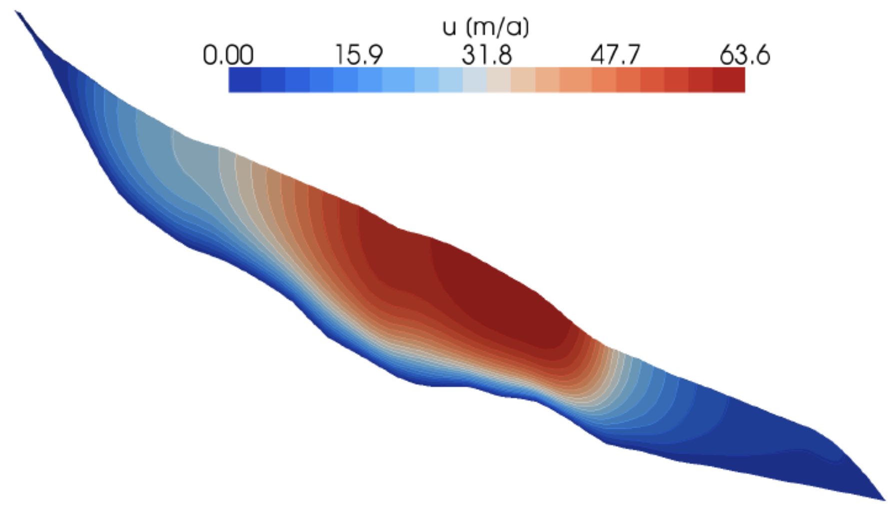
L1L2, Jacobian operator

$$\begin{bmatrix} \int_\Sigma 2(\bar{\mu}')^k \left(\dot{\varepsilon}_{1,b}^k \cdot \nabla\delta_u + \dot{\varepsilon}_{2,b}^k \cdot \nabla\delta_v\right) \dot{\varepsilon}_{1,b}^k \cdot \nabla\varphi_1 d\mathbf{x} + \\ \qquad \int_\Sigma 2\bar{\mu}^k \dot{\varepsilon}_{1,b}(\delta_u, \delta_v) \cdot \nabla\varphi_1 d\mathbf{x} + \int_\Sigma \beta\delta_u \,\varphi_1 d\mathbf{x} \\ \\ \int_\Sigma 2(\bar{\mu}')^k \left(\dot{\varepsilon}_{1,b}^k \cdot \nabla\delta_u + \dot{\varepsilon}_{2,b}^k \cdot \nabla\delta_v\right) \dot{\varepsilon}_{2,b}^k \cdot \nabla\varphi_2 d\mathbf{x} + \\ \qquad \int_\Sigma 2\bar{\mu}^k \dot{\varepsilon}_{2,b}(\delta_u, \delta_v) \cdot \nabla\varphi_2 d\mathbf{x} + \int_\Sigma \beta\delta_v \,\varphi_2 d\mathbf{x} \end{bmatrix},$$

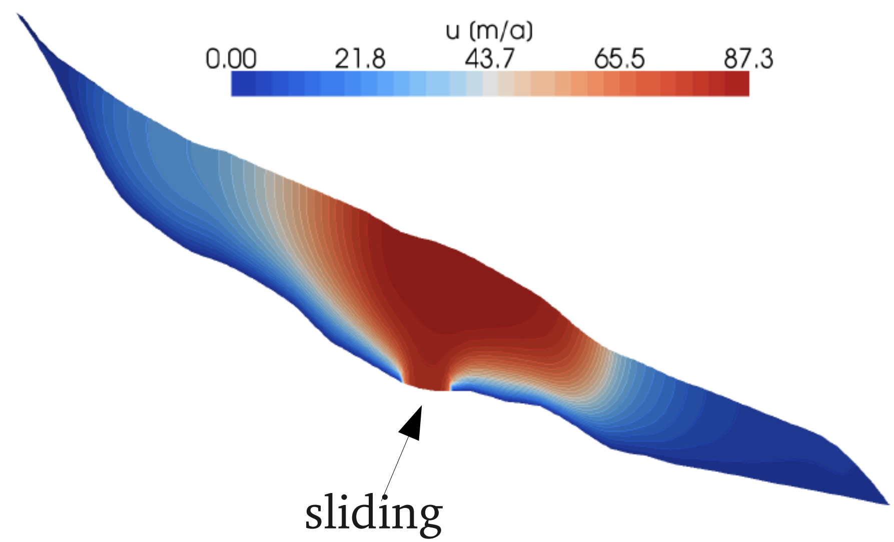
$$(\bar{\mu}')^k := \left.\frac{\partial\bar{\mu}}{\partial|\dot{\varepsilon}_b|_\parallel^2}\right|_{(\dot{\varepsilon}_b = \dot{\varepsilon}_b^k)} = \frac{1-n}{2A^2}\int_b^s \mu^k \, \frac{\left(|\tau^k|_\parallel^2 + |\tilde{\tau}|_\perp^2\right)^{1-n}}{n\,|\tau^k|_\parallel^2 + |\tilde{\tau}|_\perp^2}\, dz.$$

Test E: Glacier d'Arolla

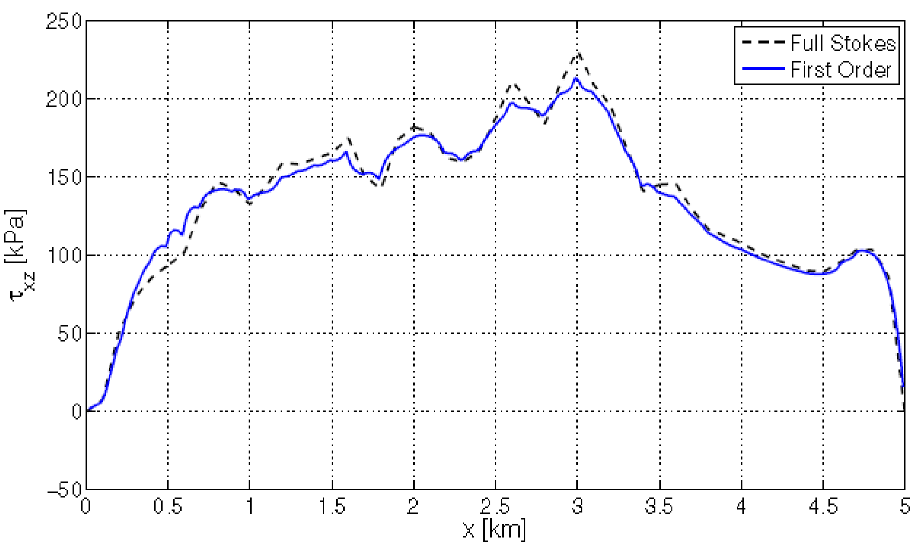
No Slip

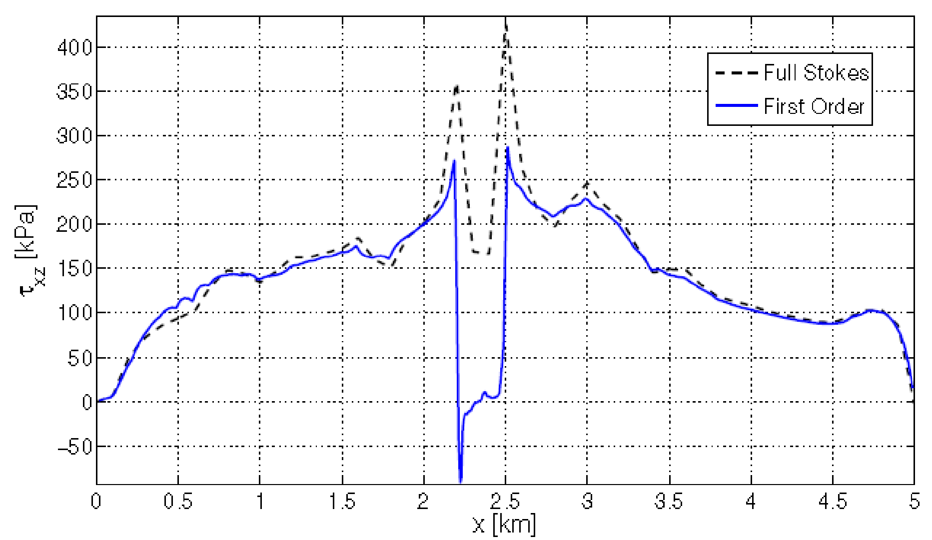


Partial sliding



Shear stress at the bedrock





Test E: Glacier d'Arolla
Comparison between linear and quadratic FE